Equivariant birational geometry of rational varieties for finite group actions

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To my cat,

José,

whose constant need for hugs has been an enormous support. Despite his comings and goings across my keyboard, the reader need not worry – my wee friend never makes any typos.

Abstract

It is nowadays well known that the finite subgroups of the Cremona group are isomorphic to the finite groups that act biregularly on rational varieties. While the question of classifying these subgroups of Cremona up to isomorphism would be answered by finding all the groups that can act on rational varieties, one can ask for a finer classification, namely up to conjugation. This leads to the beautiful theory of G-equivariant birational geometry of rational varieties.

We can outline two particularly important questions. The first is the linearization problem, that is, determining whether a subgroup G of $\mathrm{Cr}_n(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^n)$, or equivalently, the G-equivariant rationality problem. The second is the G-solidity problem. Roughly speaking, a variety is G-solid if G is not conjugate in $\mathrm{Cr}_n(\mathbb{C})$ to a group that can be decomposed into subgroups of Cremona groups of smaller rank.

In dimension two, both questions remained open after the seminal works of Blanc and Dolgachev–Iskovskikh, and we completely answer them in the present thesis. In dimension three, we further the investigations of Cheltsov–Shramov and Prokhorov on actions of the icosahedral group $\mathfrak{A}_5$ on Fano threefolds, and we treat the surprisingly open case of smooth quadrics. We provide all the G-birational models for the fixed-point-free actions of the icosahedral group $\mathfrak{A}_5$ on these varieties, and we answer the questions of linearizability and solidity for all such actions.

Lastly, we became interested in the Fano threefolds obtained by blowing up a non-hyperelliptic curve of degree six and genus three in the projective space, because their automorphism groups beautifully arise from actions on smooth plane quartics. We construct all the possible isomorphism classes of groups acting faithfully on such a variety. These threefolds also provided an opportunity to dive into the world of K-stability. In the final part of this thesis, we present how we linked their G-equivariant geometry to the existence of a Kähler–Einstein metric and produced many K-stable examples.

We will essentially consider the actions of finite groups, and, unless stated otherwise, all the work will be done over the field of complex numbers.

Lay Summary

This thesis lies in the field of Algebraic Geometry, a central area of mathematics that studies solutions to polynomial equations and the geometric structures they define, called algebraic varieties. Understanding their symmetries and maps between them beautifully brings together geometry, algebra, and group theory. We focus on a class of varieties called *rational*, their groups of symmetries, and equivariant birational maps between them. The main motivation for this work is the classification of conjugacy classes of subgroups of the Cremona groups. We first define these concepts and then show how they naturally agree and form the beautiful theory of equivariant birational geometry of rational varieties.

Acknowledgements

I am here today because my brother encouraged me to apply to the University of Edinburgh, because my dad – and Jean-Marie – made me solve my first equations, and thanks to the endless encouragement of my mom. A few lines are not enough to express all the gratitude I have for their lifelong support.

I am deeply indebted to my Ph.D. advisor, Professor Ivan Cheltsov, for his expert guidance, continuous support, and infinitely many insightful suggestions throughout the course of my doctoral studies. He was always available to discuss mathematics in ways that resonated with my vision of life: at the office, in a coffee shop, through a late Zoom call, or – literally – in a natural hot spring fueled by some non-Scottish volcano. In person or with a twelve-hour time difference, he was there to answer any question, even when I asked it for the twentieth time. With his mentorship, students no longer feel like students. They genuinely become part of the international community of algebraic geometers, and are able to touch what makes our job so great – gorgeous mathematics, friends all over the world, freedom.

I am grateful to my coauthors as well – Zhijia Zhang, Arman Sarikyan, Egor Yasinsky, Konstantin Loginov, Oliver Li, Sione Ma'u, and Joseph Malbon – for their valuable contributions and stimulating mathematical discussions.

My housemates also contributed greatly to the value of this journey. Alfred's good mood made the Scottish weather look pleasant. Amarapāla constantly offered me his invaluable insights about anything related to the language, geography, and culture of the gorgeous country that Scotland is. I never met anyone more systematically willing to help.

There was a time when many would have said that my sporting career in karate was not compatible with a mathematical future. I am immensely grateful to my former professors from *Université de Bourgogne*, Ronan Terpereau, Daniele Faenzi, and Lucy Moser-Jauslin. They gave me my chance, and I dearly hope that this thesis will prove them right.

Finally, I would like to thank the Fantastic Five – Alex, Romain, Dindon, Seb, and Mathieu. No need for a PlayStation when you have these friends to cheer you up.

As Vanya would say,

Edinburgh is beautiful.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Antoine Pinardin

Contents

Chapter 1

Introduction

"En géométrie birationnelle, les mathématiques sont heureuses."

Frédéric Mangolte

This thesis is based on five research papers. In accordance with the Academic Quality and Standards of the University of Edinburgh[1], their mathematical content has not been modified. Some necessary adjustments had to be made for consistency of the notation and definitions throughout this thesis, and some intersecting preliminary parts will be given instead in the present introduction. The papers we include are the following.

- **G-solid rational surfaces**. Pinardin (2024) — discussed in Chapter 2.
- **Linearization problem for finite subgroups of the plane Cremona group**. Pinardin, Sarikyan, and Yasinsky (2024) — discussed in Chapter 3.
- **$\mathfrak{A}_5$-equivariant geometry of quadric threefolds**. Pinardin and Zhang (2025b) — discussed in Chapter 4.
- **Abelian groups of K3 type**. Loginov, Pinardin, and Zhang (2025) — discussed in Chapter 5.
- **K-stability and space sextic curves of genus three**. Cheltsov, Li, Ma'u, and Pinardin (2024) — discussed in Chapter 6.

Unless stated otherwise, all algebraic varieties in this thesis are projective and defined over $\mathbb{C}$. The n-dimensional projective space over $\mathbb{C}$ will be denoted by $\mathbb{P}^n$, and a variety X is called *rational* if there exists a birational map $\varphi\colon X \dashrightarrow \mathbb{P}^n$. *Equivariant birational geometry* is the study of birational maps that preserve groups of symmetries. When restricted to rational varieties of dimension n, this becomes the geometrical counterpart of the study of a purely

1. https://registryservices.ed.ac.uk/academic-services/students/thesis-submission

algebraic object – the group of $\mathbb{C}$-automorphisms of the field $\mathbb{C}(x_1,\dots,x_n)$, called the Cremona group of rank n. It is isomorphic to the group of birational self-maps of the projective space, and is denoted by $\mathrm{Cr}_n(\mathbb{C})$. This correspondence between a geometrical concept and a group-theoretical one dates back to Yu. Manin and V. Iskovskikh; we will start this thesis by recalling its formalism.

1.1 G-varieties

A *G-variety* is a triple (X,G,ι), where X is a projective variety, G is a group and ι is an injective group morphism $\iota\colon G \hookrightarrow \mathrm{Aut}(X)$. A *$G$-birational map* between two G-varieties (X,G,ι) and (X',G,ι') is a birational map $\varphi\colon X \dashrightarrow X'$ such that $\varphi\iota(G)\varphi^{-1} = \iota'(G')$. If such a map φ exists, we say that (X,G,ι) and (X',G,ι') are *G-birational* to each other. Let X be a rational variety and $\varphi\colon X \dashrightarrow \mathbb{P}^n$ be a birational map. It is obvious that $\varphi\mathrm{Aut}(X)\varphi^{-1}$ is a subgroup of $\mathrm{Cr}_n(\mathbb{C})$. One can naturally ask for a converse statement, namely, given a subgroup G of $\mathrm{Cr}_n(\mathbb{C})$, whether there exists or not a rational variety $\varphi\colon X \dashrightarrow \mathbb{P}^n$ such that $\varphi^{-1}G\varphi$ is a subgroup of $\mathrm{Aut}(X)$. It is now classically known that the answer is positive for any finite group.

Theorem 1.1.1. *Let G be a finite subgroup of $\mathrm{Cr}_n(\mathbb{C})$. There exists an n-dimensional variety X and a birational map $\varphi\colon X \dashrightarrow \mathbb{P}^n$ such that $\varphi G\varphi^{-1} \subset \mathrm{Aut}(X)$. The G-variety (X,G,ι) is called a regularization of the group G.*

For a proof, the reader can refer, for example, to de Fernex and Ein (2002). Let (X,G,ι) and (X,G,ι') be two rational G-varieties. If they are G-birational, then $\Phi(X,G,\iota)$ and $\Phi(X',G,\iota')$ are subgroups of $\mathrm{Cr}_n(\mathbb{C})$ that are conjugate to each other. It induces a natural one-to-one correspondence between conjugacy classes of finite subgroups of $\mathrm{Cr}_n(\mathbb{C})$ and rational G-varieties of dimension n up to G-birational equivalence.

1.2 History of results

While the Cremona group of rank one is simply the automorphism group of the projective line, isomorphic to $\mathrm{PGL}_2(\mathbb{C})$, the problem of classifying the subgroups of $\mathrm{Cr}_n(\mathbb{C})$ becomes very involved starting from $n = 2$, and traces its origins to E. Bertini's work on involutions in $\mathrm{Cr}_2(\mathbb{C})$. Bertini identified three types of conjugacy classes, now referred to as de Jonquières, Geiser, and Bertini involutions. However, his classification was incomplete, and his proofs lacked rigour. Progress continued in 1895 with S. Kantor and A. Wiman, who provided a more detailed description of finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$, though their work was not entirely accurate either. It's only when L. Bayle and A. Beauville further developed the approach of Yu. Manin and V. Iskovskikh that the first complete proof for the classification of birational involutions was obtained; see Bayle and Beauville (2000). Later, T. de Fernex extended the classification to subgroups of prime order de Fernex (2004), while J. Blanc classified finite abelian subgroups in $\mathrm{Cr}_2(\mathbb{C})$, see Blanc (2006). The most comprehensive description of arbitrary finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$ was obtained by I. Dolgachev and V. Iskovskikh in their seminal work Dolgachev and Iskovskikh (2009).

Nevertheless, the conjugacy problem of finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$ remains open in full generality (see e.g. (Dolgachev & Iskovskikh, 2009, Section 9 *"What is left?"*)). In this Chapters 2 and 3, we settle two of its cornerstone parts, namely the questions of solidity and linearizability that we will introduce in Section 1.4 and Section 1.5.

In dimension three, classifying the subgroups of Cremona is a widely open problem. The first breakthrough was given by the classification of finite simple groups in Y. Prokhorov (2012). The author showed that only the following six non-cyclic finite simple groups belong to $\mathrm{Cr}_3(\mathbb{C})$. They are:

$$\mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7, \mathrm{PSL}_2(7), \mathrm{SL}_2(8), \mathrm{PSp}_4(3). \tag{1}$$

While the rational threefolds that have a faithful biregular action of $\mathfrak{A}_7, \mathrm{SL}_2(8)$ and $\mathrm{PSp}_4(3)$ are also classified by the author, the G-equivariant birational geometry of rational threefolds for $G \in \{\mathfrak{A}_5, \mathfrak{A}_6, \mathrm{PSL}_2(7)\}$ is currently out of reach. However, a lot of work has been achieved in the case of the icosahedral group $\mathfrak{A}_5$ in Cheltsov and Shramov (2016a) and Y. Prokhorov (2025). This group is the most fundamental one in the list (1) and carries a lot of interest from researchers in the field. Even in the case of one of the most natural Fano varieties, a smooth quadric in $\mathbb{P}^4$, the matter was not tackled. We will fill this gap it in Chapter 4, and in particular answer the questions of linearizability and solidity.

Starting from Chapter 5, our main focus will be the isomorphism classes of subgroups of $\mathrm{Cr}_3(\mathbb{C})$. In Chapter 5, we will present a new step achieved towards the full classification of finite abelian subgroups of Cr_3. We give the full list of finite abelian groups that can act on a Fano threefold while preserving a $K3$ surface in the anticanonical linear system. In fact, we conjecture that all the finite abelian groups that act on rationally connected threefolds are either included in this list, or a product of subgroups of $\mathrm{Cr}_1(\mathbb{C})$ and $\mathrm{Cr}_2(\mathbb{C})$. Finally, we will present in Chapter 6 our work on a deformation family of rational Fano varieties whose automorphism groups arise beautifully from the actions on plane quartics. These threefolds admit several equivalent descriptions, one of them being the smooth complete intersection of three divisors of bidegree $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. We will give the full list of finite groups that can act faithfully on them. Finally, we will venture to link this matter to K-stability, and produce many K-stable examples of such Fano threefolds.

1.3 Action on Mori fiber spaces

As we mentioned above in Theorem 1.3.2, any finite subgroup of $\mathrm{Cr}_n(\mathbb{C})$ can be regularized on a rational variety X. In fact, a G-equivariant version of the Minimal Model Program allows us to only consider G-Mori fiber spaces.

Definition 1.3.1. A variety Y endowed with an action of a finite group G and a G-equivariant surjective morphism $\varphi\colon Y \to Z$ with connected fibers to some variety Z with an action of G is a *G-Mori fiber space*, or G-MFS, if it has terminal singularities, all G-invariant Weil divisors on Y are $\mathbb{Q}$-Cartier divisors, the dimension of S is strictly smaller than the dimension of Y, the anticanonical class $-K_Y$ is φ-ample, and $\mathrm{rk}\,\mathrm{Cl}^G(Y) = \mathrm{rk}\,\mathrm{Cl}^G(S) + 1$. If S is a point, then Y is called a G-Fano variety.

The following well known statement is crucial for the classification conjugacy classes of subgroups of Cremona. It allows us to restrict to actions on Mori fiber spaces.

Theorem 1.3.2. *Let X be a rational variety, and F be a group that acts faithfully and biregularly on it. There exists a G-MFS which is G-birational to X.*

1.4 Equivariant solidity

As stated in Theorem 1.3.2, any finite subgroup G of $\mathrm{Cr}_n(\mathbb{C})$ is regularised on a rational G-Mori fiber space. It yields an exact sequence of the form

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

where H acts on the base of the fibration and N acts on general fibers. Since the base is rationally connected and the fibers are G-Fano varieties, this configuration splits G into components N and H acting on Mori fiber spaces of lower dimension. Such a splitting cannot be obtained when the group G only acts on G-Fano varieties up to conjugation in Cremona. Such groups were once called *primitive* by Shokurov, and are of particular importance in the classification of the subgroups of Cremona; they form its indecomposable building blocks. We present here the formal definitions of G-solidity, together with the more restrictive notions of G-rigidity and G-superrigidity.

Definition 1.4.1. Let X be a G-Fano variety.

- X is called G-solid if it is not G-birational to a G-Mori fiber space over a base of positive dimension.
- X is called G-rigid if for any G-birational map $\chi\colon X \dashrightarrow X'$, the varieties X and X' are G-isomorphic.
- X is called G-rigid if any G-birational map $\chi\colon X \dashrightarrow X'$ is a G-isomorphism.

Note that the notion of G-solidity can be extended to any rational variety. In Chapter 2, we will present the results we obtained in the paper Pinardin (2024), which gives a complete classification of G-solid rational surfaces for finite-group actions. As usual, the three-dimensional problem is much more involved and widely open. As for the linearization problem on quadrics for the actions of the icosahedral group $\mathfrak{A}_5$, we will provide the answer to solidity on those threefolds for the same actions.

1.5 The linearization problem, or equivariant rationality

The classical *rationality problem* — namely, the question of whether a given algebraic variety X of dimension n over a field $\mathbf{k}$ is birationally equivalent to the projective space $\mathbb{P}^n$ over $\mathbb{C}$ — can naturally be reformulated for algebraic varieties with a group action. Once again, this viewpoint goes back to Yu. Manin:

> *"In actuality, all of the basic results are relevant not only to rational surfaces over perfect fields, but also to a somewhat larger class of objects which we call G-surfaces. Roughly speaking, a G-surface is either a surface over a non-closed field, or a surface over an algebraically closed field, further provided with a finite group of automorphisms [...]. The main purpose of the first part of the article [...] is to show that the methods of birational classification are equally applicable to both types of G-surfaces. In particular, they permit us to classify the finite (sometimes only the abelian) subgroups of the Cremona group up to conjugacy and to obtain other information about them"*
>
> (Manin, 1967, p. 142)

Thus, given an algebraic variety X acted on by a finite group G, a natural analogue of (stable) $\mathbb{C}$-rationality problem is the problem of (stable) *G-rationality* or *linearizability of G*.

Definition 1.5.1. Let X be a smooth projective algebraic variety of dimension n equipped with a faithful action of a finite group G. We say that this action is *linearizable* or that X is *G-rational* if there is a birational map $\varphi\colon X \dashrightarrow \mathbb{P}^n$ such that $\varphi \circ g \circ \varphi^{-1}$ is an automorphism of $\mathbb{P}^n$ for all $g \in G$.

Similarly, we say that such action is *stably linearizable* or that X is *stably G-rational* if $X \times \mathbb{P}^m$ is G-rational for some $m \geqslant 0$, where the action of G on the projective space $\mathbb{P}^m$ is trivial.

For example, from this point of view, the analogue of the famous Zariski Cancellation Problem, negatively resolved in the seminal work Beauville, Colliot-Théléne, Sansuc, and Swinnerton-Dyer (1985), is the question of whether stable linearizability implies linearizability.

Despite numerous recent results in equivariant birational geometry, a brief overview of which we provide below, the problem of G-rationality remained open even in dimension 2. The main result of this paper is a *complete solution of this problem for surfaces*. To be more precise, given a finite group G, we classify all 2-dimensional G-Mori fibre spaces over $\mathbb{C}$ that are

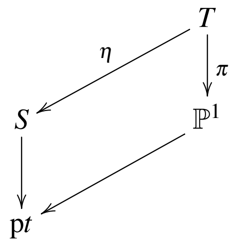
G-rational. This can be viewed as an equivariant analogue of V. Iskovskikh's criterion of $\mathbb{C}$-rationality for minimal $\overline{\mathbb{C}}$-rational surfaces (Iskovskikh, 1996, p. 642). In dimension three as well, the linearization problem is attracting considerable attention, see, e.g. Cheltsov, Marquand, Tschinkel, and Zhang (2024); Cheltsov, Marquand, Yu., and Zhang (2025); Cheltsov, Tschinkel, and Zhang (2023a, 2024); Ciurca, Tanimoto, and Tschinkel (2024). A particularly interesting class of groups is finite simple non-abelian subgroups of $\mathrm{Cr}_3(\mathbb{C})$. Once again, the most fundamental one in Prokhorov's list is the alternating group $\mathfrak{A}_5$, the smallest non-abelian simple group. It plays a significant role in birational geometry. There are only three embeddings of $\mathfrak{A}_5$ in $\mathrm{Cr}_2(\mathbb{C})$, up to conjugation. For their descriptions, see Cheltsov (2014), Dolgachev and Iskovskikh (2009) or Bannai and Tokunaga (2007). In contrast, it was shown in Krylov (2020) that there are infinitely many conjugacy classes of $\mathfrak{A}_5$ in $\mathrm{Cr}_3(\mathbb{C})$. Obtaining a classification of all such conjugacy classes is thus a difficult task and remains open. Indeed, there is a wealth of rational threefolds carrying an $\mathfrak{A}_5$-symmetry: the Segre cubic, the Igusa and Burkhardt quartic, the quintic del Pezzo threefold, etc. It is a natural question to ask about conjugation of the corresponding $\mathfrak{A}_5$-actions in $\mathrm{Cr}_3(\mathbb{C})$. In the last two decades, this has been extensively studied Avilov (2016a); Cheltsov, Przyjalkowski, and Shramov (2019), and a book Cheltsov and Shramov (2016a) was written by Cheltsov and Shramov on this topic. However, the $\mathfrak{A}_5$-equivariant geometry for one of the simplest Fano threefolds, smooth quadric threefolds, had not been addressed. In this paper, we fill this gap, answering the questions of linearizability and solidity.

1.6 Obstructions

1.6.1 Sarkisov program and birational rigidity

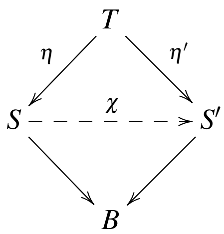
In the papers Pinardin (2024) and Pinardin et al. (2024), whose results are presented in Chapter 2 and 3, we use an extremely powerful method from birational geometry known as the *Sarkisov program*, which, in principle, allows us to answer the question of the birationality of two Mori fibre spaces. The advantage of this method is that, in dimension 2, it is very explicit; its application in higher dimensions is much more technically involved. In any case, it has enabled us to classify the finite linearizable subgroups of $\mathrm{Cr}_2(\mathbb{C})$, see Theorem 3.2.1. According to the equivariant version of the *Sarkisov program* (see e.g. (Dolgachev & Iskovskikh, 2009, Section 7)), every G-birational map between two G-Mori fibre spaces can be decomposed into a sequence of G-isomorphisms and some "elementary" G-birational maps, called *Sarkisov G-links*. These links come in four types.

Type I.



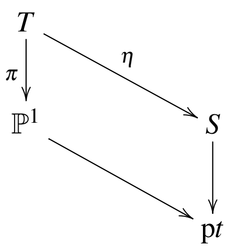
where S is a G-del Pezzo surface, η is the blow-up of a G-orbit on S, and $\pi\colon T \to \mathbb{P}^1$ is a G-conic bundle.

Type II.



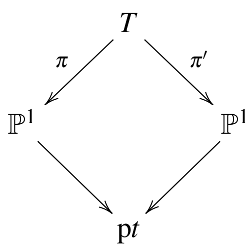
where η and η' are blow-ups of G-orbits on S and S', of lengths d and d', correspondently. In this case S and S' are G-del Pezzo surfaces if $B = \mathrm{pt}$, or S and S' are G-conic bundles if $B \simeq \mathbb{P}^1$.

Type III.



This link is the inverse to the link of type I.

Type IV.



This link is the choice of a conic bundle structure on a G-conic bundle T, which has exactly two such structures. Note that in general such a link is not represented by a biregular automorphism of T, which exchanges π and π'.

The complete classification of Sarkisov G-links between 2-dimensional G-Mori fibre spaces was obtained by V. Iskovskikh, see (Iskovskikh, 1996, Theorem 2.6) and (Dolgachev & Iskovskikh, 2009, Propositions 7.12, 7.13). We will use extensively this classification in what follows.

Convention. From now on, when talking about Sarkisov links of some specific types, we always use the notation from the diagrams above. For example, for a Sarkisov link of type II starting at a G-del Pezzo surface S, the map η always denotes the blow-up of S.

We present two classical applications of the classification of G-Sariksov links in dimension two by Iskovskikh (1996) . The first one is the modern proof of the following statement, usually called the Manin–Segre Theorem. We will need it in Chapter 2 and Chapter 3.

Theorem 1.6.1. *Let S be a G-del Pezzo surface. If $K_S^2 \leqslant 3$, then S is G-birationally rigid. If $K_S^2 = 1$, then S is G-birationally superrigid.*

The second application is the following.

Theorem 1.6.2 ((Iskovskikh, 1969, Theorem 1.6), (Cheltsov, Mangolte, Yasinsky, & Zimmermann, 2024, Theorem 2.10))**.** *Let $\pi\colon S \to \mathbb{P}^1$ be a G-conic bundle such that $K_S^2 \leqslant 0$. Then the following two assertions hold:*

(i) S is not G-birational to a smooth (weak) del Pezzo surface;

(ii) Moreover, such S is a G-birationally superrigid G-conic bundle.

We will use this result in Chapter 3. Let us now provide an overview of other obstructions to G-birational equivalence.

1.6.2 The Bogomolov–Prokhorov invariant

Consider the tower of the Cremona groups

$$\mathrm{Cr}_1(\mathbb{C}) \subset \mathrm{Cr}_2(\mathbb{C}) \subset \mathrm{Cr}_3(\mathbb{C}) \subset \mathrm{Cr}_4(\mathbb{C}) \subset \ldots,$$

where $\mathrm{Cr}_i(\mathbb{C})$ denotes the group of birational self-maps of $\mathbb{P}^i_{\mathbb{C}}$, and each embedding $\mathrm{Cr}_i(\mathbb{C}) \subset \mathrm{Cr}_{i+1}(\mathbb{C})$ is induced by adding a new variable. Two subgroups $G_1 \subset \mathrm{Cr}_n(\mathbb{C})$ and $G_2 \subset \mathrm{Cr}_m(\mathbb{C})$ are called *stably conjugate* if they are conjugate in some larger Cremona group $\mathrm{Cr}_N(\mathbb{C}) \supset \mathrm{Cr}_n(\mathbb{C}), \mathrm{Cr}_m(\mathbb{C})$, where $N \geqslant n, m$.

We say that two G-varieties (X, G) and (Y, G) are *stably birational* if there exist two integers $m, n \geqslant 0$ and a G-birational map

$$X \times \mathbb{P}^n \dashrightarrow Y \times \mathbb{P}^m,$$

where G-actions on $\mathbb{P}^n$ and $\mathbb{P}^m$ are trivial. If (X, G) is stably birational to $(\mathbb{P}^N, G)$, then we say that G is *stably linearizable*. The following question can be viewed as an analogue of Zariski Cancellation Problem in our setting:

Question 1.6.3. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a stably linearizable finite subgroup. Is G linearizable?

As was observed in Bogomolov and Prokhorov (2013), for a smooth projective G-variety X, the cohomology group $\mathrm{H}^1(G,\mathrm{Pic}(X))$ is a G-birational invariant (in the context of rationality questions over non-closed fields this observation goes back to Yu. Manin). Moreover, one has the following.

Theorem 1.6.4 ((Bogomolov & Prokhorov, 2013, Corollary 2.5.1))**.** *If (X,G) and (Y,G) are projective, smooth, and stably birational, then*

$$\mathrm{H}^1(G,\mathrm{Pic}(X)) \simeq \mathrm{H}^1(G,\mathrm{Pic}(Y)).$$

Corollary 1.6.5 ((Bogomolov & Prokhorov, 2013, Corollary 2.5.2))**.** *If (X,G) is stably linearizable, then $\mathrm{H}^1(H,\mathrm{Pic}(X)) = 0$ for any subgroup $H \subset G$.*

If the latter condition holds, the G-variety X is said to be H^1*-trivial*. It turns out that in some cases, the invariant $\mathrm{H}^1(G,\mathrm{Pic}(X))$ can be computed in terms of G-fixed locus:

Theorem 1.6.6. *(Bogomolov & Prokhorov, 2013, Theorem 1.1, Corollary 1.2) Let a finite cyclic group G of prime order p act on a non-singular projective rational surface X. Assume that G fixes (point-wise) a curve of genus $g > 0$. Then*

$$\mathrm{H}^1(G,\mathrm{Pic}(X)) \simeq (\mathbb{Z}/p\mathbb{Z})^{2g}.$$

Moreover, the following are equivalent:

1. *G is H^1-trivial;*
2. *(X,G) is linearizable;*
3. *(X,G) is stably linearizable.*

For an independent proof, the reader can also refer to Shinder (2016). In particular, de Jonquières, Bertini and Geiser involutions are not stably linearizable: they all fix a curve of positive genera.

In Y. Prokhorov (2015), Prokhorov further applies the invariant introduced by Bogomolov and himself and indicates where to look for negative answers to Question 1.6.3. He shows, for example, that a del Pezzo surface S acted on by a finite group G with $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$, is H^1-trivial if and only if (i) either $K_S^2 \geqslant 5$, (ii) or S is the quartic

$$x_1^2 + \omega_3 x_2^2 + \omega_3^2 x_3^2 + x_4^2 = x_1^2 + \omega_3^2 x_2^2 + \omega_3 x_3^2 + x_5^2 = 0$$

in $\mathbb{P}^4$, acted on by the group $G \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ generated by two automorphisms

$$(x_1,x_2,x_3,x_4,x_5) \mapsto (x_2,x_3,x_1,\omega_3 x_4,\omega_3^2 x_5), \quad (x_1,x_2,x_3,x_4,x_5) \mapsto (x_1,x_3,x_2,-x_5,x_4),$$

where $\omega_3 = \exp(2\pi i/3)$. Our Main Theorem below shows that only a few groups in these two cases (i) and (ii) are linearizable. Constructions of *stable* linearizability in the remaining non-linearizable cases are sporadic and remain an open problem even in dimension 2. For instance, as of the time of writing, it is unknown whether the group $G \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ mentioned above is stably linearizable. Some of these constructions are mentioned below, see Remarks 3.4.4, 3.4.15, and 3.4.16.

1.6.3 The Burnside formalism

Another very recent technique for distinguishing birational actions of finite groups is the *Burnside group formalism*, introduced in the work Kresch and Tschinkel (2022b) of A. Kresch and Yu. Tschinkel and generalizing the *birational symbols groups* of Kontsevich, Pestun, and Tschinkel (2019). Here, we will limit ourselves to a rough sketch of this approach.

In Kresch and Tschinkel (2022b), the *symbols group* $\mathrm{Symb}_n(G)$ was defined as the free abelian group generated by *symbols* $(H, R \circlearrowright \mathbb{C}, \beta)$. Here, $H \subset G$ is an abelian subgroup, $R \subset \mathrm{C}_G(H)/H$ is a subgroup, where $\mathrm{C}_G(H)$ denotes the centralizer of H, $\mathbb{C}/\mathbb{C}$ is a finitely generated extension of transcendence degree $d \leqslant n$ faithfully acted on by R, and $\beta = (b_1, \ldots, b_{n-d})$ is a sequence of nontrivial characters of H, generating $\mathrm{Hom}(H, \mathbb{C}^*)$. The quotient of $\mathrm{Symb}_n(G)$ by some involved *conjugation* and *blow-up relations* (Kresch & Tschinkel, 2022b, Section 4) gives the *equivariant Burnside group* $\mathrm{Burn}_n(G)$.

Now, let (X, G) be a smooth n-dimensional projective variety with a generically free action of G. By (Reichstein & Youssin, 2000, Theorem 3.2) or (Hassett, Kresch, & Tschinkel, [2021] ©2021, Section 7.2), the action of G can always be brought to a *standard form* via equivariant blow-ups. This means that there is a G-invariant simple normal crossing divisor $\Delta \subset X$ such that G acts freely on $X \setminus \Delta$, and for every $g \in G$ and every irreducible component D of Δ one has either $g(D) = D$ or $g(D) \cap D = \varnothing$. Let $\{D_j\}_{j \in \mathscr{J}}$ be the set of irreducible components with non-trivial (and hence cyclic) stabilizers $H_j \subset G$, considered up to conjugation in G. For each $j \in \mathscr{J}$, the elements of G that do not move D_j to another component of Δ, give rise to a subgroup $R_j \subset \mathrm{C}_G(H_j)/H_j$. Consider the subset $\mathscr{I} \subset \mathscr{J}$ corresponding to those divisors, together with the respective R_j-action, that cannot be obtained via equivariant blow-ups of any standard model of any G-variety (such divisors were called *incompressible*). Finally, let b_j be the character of H_j in the normal bundle to D_j. The formal sum

$$[X \circlearrowleft G] = \sum_{i \in \mathscr{I}} (H_i, R_i \circlearrowright \mathbb{C}(D_i), b_i), \tag{2}$$

viewed as an element of $\mathrm{Burn}_n(G)$, turns out to be a well-defined G-birational invariant (Kresch & Tschinkel, 2022b, Theorem 5.1).

There exist different versions of symbols groups and corresponding Burnside-type obstructions, see e.g. (Tschinkel, Yang, & Zhang, 2023, Section 3) for an overview. The first applications of this formalism yields non-linearizable cyclic actions on certain cubic fourfolds (Hassett et al., [2021] ©2021, 6); a new proof (Hassett et al., [2021] ©2021, 7.6) of non-linearizability of minimal D_6-action on the sextic del Pezzo surface (this is an Iskovskikh's example 3.4.13); non-conjugacy[2] of certain intransitive and imprimitive subgroups of $\mathrm{PGL}_3(\mathbb{C})$ and $\mathrm{PGL}_4(\mathbb{C})$ in $\mathrm{Cr}_2(\mathbb{C})$ and $\mathrm{Cr}_3(\mathbb{C})$, respectively (by showing that the corresponding actions give different classes in Burn_2 and Burn_3), see (Kresch & Tschinkel, 2022c, Sections 10-11) and (Tschinkel et al., 2023, Sections 7-8). Furthermore, there are examples of non-linearizable actions on some 3-dimensional quadrics (Tschinkel et al., 2023, Section 9), and classification (obtained through combinations of various techniques) of non-linearizable actions on some prominent threefolds, such as the Segre cubic, the Burkhardt quartic, and some singular cubic threefolds Cheltsov, Marquand, et al. (2024); Cheltsov, Tschinkel, and Zhang (2023b, 2024).

Some other equivariant birational invariants were recently proposed by L. Esser (the dual complex, see Esser (2024)), T. Ciurca, S. Tanimoto and Yu. Tschinkel (an equivariant version of the formalism of intermediate Jacobian torsor obstructions, see Cheltsov, Tschinkel, and Zhang (2024); Ciurca et al. (2024)), and by J. Blanc, I. Cheltsov, A. Duncan and Yu. Prokhorov (the Amitsur subgroup, see Blanc, Cheltsov, Duncan, and Prokhorov (2023)). We refer the reader to these works for details.

1.6.4 Noether-Fano inequalitiy

The cornerstone of birational rigidity and birational solidity is the classical Noether–Fano inequality, which reveals a close connection between canonical singularities of log pairs and the existence of birational maps between Mori fibre spaces. In fact, the decomposition theorem for G-Sarkisov links from Iskovskikh (1996) described above is a consequence of Noether–Fano inequality. We will recall it in the equivariant setting, as is in (Cheltsov & Shramov, 2016a, Theorem 3.2.6). Let X be a projective variety with at most klt singularities and G a finite subgroup of $\mathrm{Aut}(X)$. We use the language of the (equivariant) minimal model program; see, e.g., (Cheltsov & Shramov, 2016b, I.2). Throughout the paper, a log pair $(X, \mathscr{M}_X)$ refers to a pair consisting of X with a non-empty mobile G-invariant linear system $\mathscr{M}_X$ on X consisting of $\mathbb{Q}$-Cartier divisors. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities, and

$$\mathscr{M}^{\widetilde{X}} := \pi^*(K_X + \mathscr{M}_X) - K_{\widetilde{X}}.$$

For a prime divisor $D \subset \mathrm{Supp}(\mathscr{M}^{\widetilde{X}})$, the *log discrepancy* of the log pair $(X, \mathscr{M}_X)$ at D is defined as the rational number

$$a(X, \mathscr{M}_X; D) = 1 - \mathrm{mult}_D(\mathscr{M}^{\widetilde{X}}).$$

2. Note that this cannot be approached via the Bogomolov–Prokhorov invariant, which vanishes for linear actions.

Let $p \in X$ be a point. We say that $(X, \mathscr{M}_X)$ is *canonical (resp. log-canonical, klt)* at p if for any prime divisor E on $\widetilde{X}$ such that $p \in \pi(E)$, we have $a(X, \mathscr{M}_X; E) \geq 1$ (resp. ≥ 0, > 0). The *non-canonical (resp. non-log-canonical, non-klt) locus* of $(X, \mathscr{M}_X)$ is the union of points where $(X, \mathscr{M}_X)$ is not canonical (resp. not log-canonical, not klt).

An irreducible subvariety $Z \subset X$ is said to be a *center of non-canonical (resp. non-log-canonical, non-klt) singularities* of $(X, \mathscr{M}_X)$ if there exists a resolution $\pi : \widetilde{X} \to X$ and a prime divisor E on $\widetilde{X}$ such that $\pi(E) = Z$ and $a(X, \mathscr{M}_X; E) < 1$ (resp. < 0, ≤ 0). For simplicity, we also refer to it as *a non-canonical (resp. non-log-canonical, non-klt) center*.

Theorem 1.6.7 (Noether–Fano inequality)**.** *Let X be a Fano variety with terminal singularities, G a finite subgroup of $\mathrm{Aut}(X)$ such that $\mathrm{rk}(\mathrm{Cl}^G(X)) = 1$. Assume that there exists a G-equivariantly birational map $\chi : X \dashrightarrow V$, where V is a variety with a generically free G-action such that one of the following holds:*

1. *either V is also a Fano variety with terminal singularities such that $\mathrm{rk}(\mathrm{Cl}^G(V)) = 1$;*
2. *or there exists a G-equivariant morphism $V \to Z$ with connected fibres such that its general fibre is a Fano variety, and Z is a normal projective variety with $\dim(V) > \dim(Z) > 0$.*

In the former case, let $\mathscr{M}_X$ be the strict transform on X of the linear system $|-nK_V|$ for $n \gg 0$. In the latter case, let $\mathscr{M}_X$ be the strict transform on X of the linear system $|H_V|$, where H_V is the pullback on V of a very ample divisor on Z whose class in $\mathrm{Pic}(Z)$ is G-invariant. Let $\lambda \in \mathbb{Q}$ be such that $\lambda \mathscr{M}_X \sim_{\mathbb{Q}} -K_X$. Then, if χ is not biregular, the log pair $(X, \lambda \mathscr{M}_X)$ has non-canonical singularities.

Theorem 1.6.7 will be crucial in Chapter 4. We will also make extensive use of the following statements related to singularities of pairs. The α-invariant is a number associated to X which corresponds to the *global log-canonical threshold* introduced in Tian (1987) in a different language. When X is a Fano variety with $\dim(X) \geq 2$, the *G-equivariant α-invariant* of X is the number

$$\alpha_G(X) = \sup \left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the pair } (X, \lambda D) \text{ is log-canonical for any} \\ G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \end{array} \right\}.$$

We compute this invariant for some G-surfaces in Sections 4.3 and 4.4. We will use a few results about singularities.

Theorem 1.6.8 ((Pukhlikov, 2013, Theorem 2.1))**.** *Let X be a threefold and $\mathscr{M}_X$ a non-empty mobile linear system on X. If a smooth point $p \in X$ is a non-canonical center of the pair $(X, \lambda \mathscr{M}_X)$ for some positive rational number λ, and D_1, D_2 are two general elements in $\mathscr{M}_X$, then*

$$\mathrm{mult}_p(D_1 \cdot D_2) > \frac{4}{\lambda^2}.$$

Theorem 1.6.8 is essentially a corollary of the following theorem due to Corti and the inversion of adjunction; see also (Cheltsov & Shramov, 2016b, Section 2.5).

Theorem 1.6.9 ((Corti, 2000, Theorem 3.1))**.** *Let S be a surface and $\mathscr{M}_S$ a non-empty mobile linear system on S. If a smooth point $p \in S$ is a non-log-canonical center of the pair $(S, \lambda\mathscr{M}_S)$ for some positive rational number λ, and D_1, D_2 are two general elements in $\mathscr{M}_S$, then*

$$\mathrm{mult}_p(D_1 \cdot D_2) > \frac{4}{\lambda^2}.$$

In many situations, Theorem 1.6.8 gives us a desired bound. However, in certain situations (e.g., in Propositions 4.5.22 and 4.6.12), a sharper result is needed:

Theorem 1.6.10 (Demailly and Pham (2014))**.** *Let S be a smooth surface, $p \in S$ a point, and $\mathscr{M}_S$ a non-empty mobile linear system on S. Assume that p is a non-log-canonical center of the log pair $(S, \lambda\mathscr{M}_S)$ with some positive rational number λ. Let $m = \mathrm{mult}_p(\mathscr{M}_S)$. Then for two general elements $D_1, D_2 \in \mathscr{M}_S$, we have*

$$\mathrm{mult}_p(D_1 \cdot D_2) > \frac{m^2}{\lambda^2(m-1)}.$$

We will also use the following technical observation.

Remark 1.6.11 ((Cheltsov, Sarikyan, & Zhuang, 2023, Remark 3.6))**.** Let X be a threefold with terminal singularities, $p \in X$ a smooth point, $\mathscr{M}_X$ a mobile linear system, and $\lambda \in \mathbb{Q}_{>0}$. If p is a non-canonical center of the log pair $(X, \mathscr{M}_X)$, then p is a non-log-canonical center of the log pair $(X, \frac{3}{2}\mathscr{M}_X)$.

The Nadel vanishing theorem will give us bounds of the size of 0-dimensional non-canonical centers.

Theorem 1.6.12 ((Lazarsfeld, 2003, Theorem 9.4.8))**.** *Let X be a projective variety with at most klt singularities, D an effective $\mathbb{Q}$-divisor on X, L a Cartier divisor such that $K_X + D + A \sim_{\mathbb{Q}} L$ for some ample divisor A, and $\mathscr{I}(X, D)$ the multiplier ideal sheaf of D. Then*

$$H^i(X, \mathscr{O}_X(L) \otimes \mathscr{I}(X, D)) = 0 \text{ for } i \geq 1.$$

Lastly, we introduce some terminology. For a G-invariant subvariety $Z \subset X$, we say that Z is *G-irreducible* if G acts transitively on the irreducible components of Z. Let H be a general hyperplane section on X. We denote by $|nH - Z|$ the linear system consisting of degree n hyperplane sections on X passing through Z. Often, we refer to this as the linear system $|nH - Z|$, although Z is not necessarily a divisor.

1.7 Group theory

This Section is entirely devoted to auxiliary results from the theory of finite groups. These results are either elementary (nevertheless, we provide proofs for the reader's convenience) or pertain to the classical representation theory.

Notations 1.7.1. *Throughout our paper, we use the following standard notations:*

— $\omega_n = \exp(2\pi i/n)$, *where* $n \in \mathbb{N}$*;*

— $\mathbb{Z}_n$ *is a cyclic group of order* n*;*

— $\mathrm{V}_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ *is Klein's Vierergruppe;*

— Q_8 *is the quaternion group;*

— D_n *is the dihedral group of order* $2n$*;*

— $\mathfrak{S}_n$ *is the permutation group of degree* n*;*

— $\mathfrak{A}_n$ *is the alternating group of degree* n*;*

— $\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$ *is the holomorph of a group* G*;*

— $\mathrm{F}_5 = \mathrm{Hol}(\mathbb{Z}_5)$ *is the Frobenius group of order 20;*

— $\mathrm{D}(A)$ *is the generalized dihedral group over an abelian group* A*;*

— $A_\bullet B$ *is an extension (not necessarily split) of* B *with a normal subgroup* A*;*

— $A \times_Q B$ *is a fibred product of* A *and* B *over their common homomorphic image* Q*; see Section 1.7.3 for more details;*

— $G \wr \mathfrak{S}_n$ *is the wreath product, i.e. the semi-direct product* $G^n \rtimes \mathfrak{S}_n$*, where* $\mathfrak{S}_n$ *acts on* G^n *by permuting the factors;*

— $\mathrm{C}_G(H)$ *is the centralizer of* $H \subset G$ *in* G*;*

— $\mathrm{N}_G(H)$ *is the normalizer of* $H \subset G$ *in* G*;*

— $\mathrm{Z}(G)$ *is the centre of a group* G*;*

— *Inn* (G) *is the inner automorphism group of a group* G*.*

1.7.1 Klein's classification

First, we recall the following classical result of Felix Klein.

Proposition 1.7.2 (Klein (2019))**.** *Every finite subgroup of* $\mathrm{PGL}_2(\mathbb{C})$ *is isomorphic to* $\mathbb{Z}_n$*,* D_n $($*where* $n \geqslant 1)$*,* $\mathfrak{A}_4$*,* $\mathfrak{S}_4$ *or* $\mathfrak{A}_5$*. Moreover, there is only one conjugacy class for each of these groups.*

In what follows, we will also need the following more general result, which describes the algebraic subgroups of $\mathrm{PGL}_2(\mathbb{C})$.

Theorem 1.7.3 ((Kaplansky, 1957, p. 31), (Nguyen, van der Put, & Top, 2008, Theorem 1))**.** *Up to conjugation, every algebraic subgroup of* $\mathrm{PGL}_2(\mathbb{C})$ *is one of the following:*

1. $\mathrm{PGL}_2(\mathbb{C})$.
2. *A finite subgroup from the Klein's list:* $\mathbb{Z}_n, \mathrm{D}_n, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5$.

3. *The image under the natural projection* $\pi\colon \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{PGL}_2(\mathbb{C})$ *of the Borel (i.e. maximal solvable) subgroup*

$$\mathrm{B} = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{C}^*, b \in \mathbb{C} \right\} \subset \mathrm{SL}_2(\mathbb{C}).$$

4. *The image under the natural projection* $\pi\colon \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{PGL}_2(\mathbb{C})$ *of the infinite dihedral subgroup*

$$\mathrm{D}_\infty = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & -b \\ b^{-1} & 0 \end{bmatrix} : b \in \mathbb{C}^* \right\}.$$

Remark 1.7.4. In case (3), let $\overline{\mathrm{B}} = \pi(\mathrm{B})$, and consider the unipotent (abelian) subgroup

$$\overline{\mathrm{U}} = \left\{ \overline{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}} : b \in \mathbb{C} \right\} \subset \overline{\mathrm{B}} = \left\{ \overline{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}} : a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$

The elements of $\overline{\mathrm{B}}$ can be viewed as affine automorphisms of $\mathbb{P}^1$, locally given by $z \mapsto az + b$. The map which sends every such automorphism to $a \in \mathbb{C}^*$ is a well-defined group homomorphism, which induces a short exact sequence

$$1 \longrightarrow \overline{\mathrm{U}} \longrightarrow \overline{\mathrm{B}} \longrightarrow \mathbb{C}^* \longrightarrow 1. \tag{3}$$

In the case (4), letting $\overline{\mathrm{D}}_\infty = \pi(\mathrm{D}_\infty)$, we observe that the diagonal matrices $\mathrm{diag}\{a, a^{-1}\}$, where $a \in \mathbb{C}^*$, constitute the connected component of the identity D_∞° and are mapped by π to a group isomorphic to $\mathbb{C}^*$. The other matrices are mapped to involutions. Furthermore, one has $\overline{\mathrm{D}}_\infty \simeq \mathbb{C}^* \rtimes \mathbb{Z}_2$, where the action is by inversion: $t \mapsto t^{-1}$, i.e. $\overline{\mathrm{D}}_\infty \simeq \mathrm{Hol}(\mathbb{C}^*)$.

Let us fix the following presentations (Huppert, 1967, §19):

- $\mathbb{Z}_n \simeq \langle r \mid r^n = \mathrm{id} \rangle$.
- $\mathrm{D}_n \simeq \langle r, b \mid r^n = b^2 = (rb)^2 = \mathrm{id} \rangle$.
- $\mathfrak{A}_4 \simeq \langle a, b, c \mid a^2 = b^2 = c^3 = \mathrm{id},\ cac^{-1} = ab = ba,\ cbc^{-1} = a \rangle$. Note that $\langle a, b \rangle \simeq \mathrm{V}_4$ is the derived subgroup of $\mathfrak{A}_4$ and one has $\mathfrak{A}_4 \simeq \langle a, b \rangle \rtimes \langle c \rangle$.
- $\mathfrak{S}_4 \simeq \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^2 = \mathrm{id},\ cac^{-1} = dad = ab = ba,\ cbc^{-1} = a,\ bd = db,\ dcd = c^{-1} \rangle$. Here, one finds a unique copy of $\mathfrak{A}_4 \simeq \langle a, b, c \rangle$ inside, which is the derived subgroup of $\mathfrak{S}_4$, and one has $\mathfrak{S}_4 \simeq \langle a, b, c \rangle \rtimes \langle d \rangle$.
- $\mathfrak{A}_5 \simeq \langle e, f \mid e^5 = f^2 = (ef)^3 = \mathrm{id} \rangle$.

Notations 1.7.5. *Let us fix the following identifications (see e.g. (Faber, 2023, Theorem C)):*

$$r \mapsto \mathbb{R}_n = \begin{bmatrix} 1 & 0 \\ 0 & \omega_n \end{bmatrix},\ a \mapsto \mathrm{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ b \mapsto \mathrm{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ c \mapsto \mathrm{C} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix},$$

$$d \mapsto \mathrm{D} = \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix},\ e \mapsto \mathrm{E} = \begin{bmatrix} \omega_5 & 0 \\ 0 & 1 \end{bmatrix},\ f \mapsto \mathrm{F} = \begin{bmatrix} 1 & 1-\omega_5-\omega_5^{-1} \\ 1 & -1 \end{bmatrix}.$$

Then these identifications define embeddings of $\mathbb{Z}_n$, D_n, $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ into $\mathrm{PGL}_2(\mathbb{C})$.

The following description of orbits on $\mathbb{P}^1$ is classical:

Proposition 1.7.6 ((Springer, 1977, 4.4))**.** *Let G be a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$. One has the following.*

1. *If $G \simeq \mathbb{Z}_n$ then it has 2 fixed points on $\mathbb{P}^1$, namely $[1:0]$ and $[0:1]$, and any other point generates an orbit of length n.*
2. *If $G \simeq \mathrm{D}_n$, then it has one orbit of length 2, namely $[1:0]$ and $[0:1]$, and one orbit of length n generated by $[1:1]$. Any other point generates an orbit of length $2n$.*
3. *If $G \simeq \mathfrak{A}_4$ then it has two orbits of length 4 and one orbit of length 6. All other orbits are of length 12.*
4. *If $G \simeq \mathfrak{S}_4$ then it has one orbit of length 6, one orbit of length 8 and one orbit of length 12. All other orbits are of length 24.*
5. *If $G \simeq \mathfrak{A}_5$ then it has one orbit of length 12, one orbit of length 20, and one orbit of length 30. All other orbits are of length 60.*

1.7.2 Blichfeldt's classification

Since we are interested in linearization of finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$, let us first recall the classification of finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$, which is essentially due to H. Blichfeldt.

Definition 1.7.7 (Blichfeldt (1917))**.** We call a subgroup $\iota\colon G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ *intransitive* if the representation ι is reducible, and *transitive* otherwise. Further, a transitive group G is called *imprimitive* if there is a decomposition $\mathbb{C}^n = \bigoplus_{i=1}^m V_i$ into a direct sum of subspaces and G transitively acts on the set $\{V_i\}$. A transitive group G is called *primitive* if there is no such decomposition. Finally, we say that $G \subset \mathrm{PGL}_n(\mathbb{C})$ is (in)transitive or (im)primitive if its lift to $\mathrm{GL}_n(\mathbb{C})$ is such a group.

So, intransitive subgroups of $\mathrm{PGL}_3(\mathbb{C})$ come in two types:

- I_1 The representation of G in $\mathrm{GL}_3(\mathbb{C})$ is a direct sum of three 1-dimensional representations. In other words, G fixes 3 non-collinear points on $\mathbb{P}^2$ and hence $G \simeq \mathbb{Z}_n \times \mathbb{Z}_m$, where $n, m \geqslant 1$, is a diagonal abelian group.
- I_2 The representation of G in $\mathrm{GL}_3(\mathbb{C})$ is a direct sum of 1-dimensional and 2-dimensional representations. In other words, G fixes a point $p \in \mathbb{P}^2$, and hence there is an embedding $G \hookrightarrow \mathrm{GL}(T_p\mathbb{P}^2) \simeq \mathrm{GL}_2(\mathbb{C})$. Obviously, every finite subgroup of $\mathrm{GL}_2(\mathbb{C})$ gives rise to an intransitive subgroup of $\mathrm{PGL}_3(\mathbb{C})$.

Assume that G is imprimitive. Then $\mathbb{C}^3 = V_1 \oplus V_2 \oplus V_3$, where $V_1 \simeq V_2 \simeq V_3 \simeq \mathbb{C}$ and G acts transitively on the set $\{V_1, V_2, V_3\}$. Hence G fits into the short exact sequence

$$1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1$$

where $B \simeq \mathbb{Z}_3$ or $B \simeq \mathfrak{S}_3$, and $A \subset \mathrm{PGL}_3(\mathbb{C})$ is an intransitive subgroup of type I_1.

Finally, there are 6 primitive subgroups of $\mathrm{PGL}_3(\mathbb{C})$.

GAP	Order	Isomorphism class	Comments
Primitive subgroups			
$[36,9]$	36	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$	subgroup of the Hessian group
$[72,41]$	72	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathrm{Q}_8$	subgroup of the Hessian group
$[216,153]$	216	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathrm{SL}_2(\mathbb{F}_3)$	the Hessian group
$[60,5]$	60	$\mathfrak{A}_5$	the simple icosahedral group
$[168,42]$	168	$\mathrm{PSL}_2(\mathbb{F}_7)$	the simple Klein group
$[360,118]$	360	$\mathfrak{A}_6$	the simple Valentiner group

1.7.3 Goursat's lemma

Let Γ_1 and Γ_2 be two finite groups. Finite subgroups of the direct product $\Gamma_1 \times \Gamma_2$ can be determined using the classical *Goursat's lemma*. Recall that the *fibre product* of two groups G_1 and G_2 over a group Q is defined as

$$G_1 \times_Q G_2 = \{(g_1, g_2) \in G_1 \times G_2 \colon \alpha(g_1) = \beta(g_2)\},$$

where $\alpha\colon G_1 \to Q$ and $\beta\colon G_2 \to Q$ are surjective homomorphisms. Note that the data defining $G_1 \times_Q G_2$ is not only the groups G_1, G_2 and Q but also the homomorphisms α, β.

Lemma 1.7.8 (Goursat's lemma, (Goursat, 1889, p. 47))**.** *Let Γ_1 and Γ_2 be two finite groups. There is a bijective correspondence between subgroups $G \subseteq \Gamma_1 \times \Gamma_2$ and 5-tuples $\{G_1, G_2, H_1, H_2, \varphi\}$, where G_1 is a subgroup of Γ_1, H_1 is a normal subgroup of G_1, G_2 is a subgroup of Γ_2, H_2 is a normal subgroup of G_2, and $\varphi\colon G_1/H_1 \xrightarrow{\sim} G_2/H_2$ is an isomorphism. Namely, the group corresponding to this 5-tuple is*

$$G = \{(g_1, g_2) \in G_1 \times G_2 \colon \varphi(g_1 H_1) = g_2 H_2\}.$$

Conversely, let $G \subseteq \Gamma_1 \times \Gamma_2$ be a subgroup. Denote by $p_1\colon \Gamma_1 \times \Gamma_2 \to \Gamma_1$ and $p_2\colon \Gamma_1 \times \Gamma_2 \to \Gamma_2$ the natural projections, and set $G_1 = p_1(G)$, $G_2 = p_2(G)$. Set

$$H_1 = \ker p_2|_G = \{(g_1, \mathrm{id}) \in G,\ g_1 \in \Gamma_1\},\ \ H_2 = \ker p_1|_G = \{(\mathrm{id}, g_2) \in G,\ g_2 \in \Gamma_2\},$$

whose images by p_1 and p_2 define normal subgroups of G_1 and G_2, respectively (denoted the same). Let $\pi_1\colon G_1 \to G_1/H_1$ and $\pi_2\colon G_2 \to G_2/H_2$ be the quotient homomorphisms. The map $\varphi\colon G_1/H_1 \to G_2/H_2$, $\varphi(g_1 H_1) = g_2 H_2$, where $g_2 \in \Gamma_2$ is any element such that $(g_1, g_2) \in G$, is an isomorphism. Furthermore, $G = G_1 \times_Q G_2$, where $Q = G_1/H_1$, $\alpha = \pi_1$, and $\beta = \varphi^{-1} \circ \pi_2$.

Lemma 1.7.9. *In the above notation, any subgroup $G \subseteq \Gamma_1 \times \Gamma_2$ fits into the short exact sequence*

$$1 \longrightarrow H_1 \times H_2 \longrightarrow G \longrightarrow Q \longrightarrow 1$$

Proof. Indeed, the restriction of the homomorphism $\alpha \times \beta \colon G_1 \times G_2 \to Q \times Q$ to G has the kernel $H_1 \times H_2$ and the image is isomorphic to $\{(t,t) \in Q \times Q\} \simeq Q$. □

Remark 1.7.10. We will often use explicit generators to describe a fibre product. They are given as follows. Let S be a subset of G_1 such that $\alpha(S) = Q$. Then the group $G_1 \times_Q G_2$ is generated by $\ker\alpha \times \ker\beta$, and $\{(g_1, g_2), g_1 \in S, \alpha(g_1) = \beta(g_2)\}$.

1.7.4 Actions on sets

We will use several elementary lemmas about group actions on a set.

Lemma 1.7.11. *Let A and B be two finite groups, and $G \subset A \times B$ be a subgroup such that the natural projection $G \to A$ is surjective. Assume that A acts on a set X and B acts on a set Y. Consider the induced action of G on $X \times Y$. Then the length of any orbit of G in $X \times Y$ is divisible by the length of some orbit of A on X.*

Proof. Pick a point $(x_0, y_0) \in X \times Y$ and consider its G-orbit $\Sigma = \{(ax_0, by_0) \colon (a,b) \in G\}$. Consider the projection $p_X \colon X \times Y \to X$. We claim that the set $\Sigma_X = p_X(\Sigma) \subset X$ is an A-orbit. Firstly, it is clearly A-invariant: taking any $x \in \Sigma_X$ there exists $y \in Y$ such that $(x,y) \in \Sigma$. By our assumption, for any $a \in A$ there exists $b \in B$ such that $(a,b) \in G$. Then $(ax, by) \in \Sigma$ and therefore $ax \in \Sigma_X$. Secondly, Σ_X is clearly a minimal A-invariant set. For if there is a proper A-invariant subset $\Sigma'_X \subsetneq \Sigma_X$, then taking two elements $x' \in \Sigma'_X$, $x \in \Sigma_X \setminus \Sigma'_X$ and their lifts $(x', y') \in \Sigma$, $(x,y) \in \Sigma$, there exists an element $(a,b) \in G$ such that $(ax', by') = (x,y)$. In particular, $x = ax' \notin \Sigma'_X$; but this contradicts Σ'_X being A-invariant.

It remains to show that for each $x_1 \in \Sigma_X$, the cardinality of the fibre $p_X^{-1}(x_1) \cap \Sigma$ is the same and equals to the cardinality of $p_X^{-1}(x_0) \cap \Sigma$. Indeed, there exists $a \in A$ such that $ax_0 = x_1$. Pick any lift $(a,b) \in G$ and consider the map of finite sets

$$p_X^{-1}(x_0) \cap \Sigma \to p_X^{-1}(x_1) \cap \Sigma, \ (x_0, y) \mapsto (ax_0, by) = (x_1, by),$$

which is obviously well defined and injective. The same holds for the map $p_X^{-1}(x_1) \cap \Sigma \to p_X^{-1}(x_0) \cap \Sigma$ which sends (x_1, y) to $(a^{-1}x_1, b^{-1}y) = (x_0, b^{-1}y)$. This finishes the proof. □

Lemma 1.7.12. *Let G be a finite group and $H \subset G$ be a normal subgroup of finite index $[G:H]$. Assume that G acts on a set X and let Σ be an orbit of G. Write $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_n$ as the disjoint union of H-orbits Σ_i. Then all Σ_i have the same length and n divides $[G:H]$. In particular, the length of Σ is always divisible by the length of some orbit of H.*

Proof. Indeed, the quotient group G/H acts on the space of orbits Σ/H, and this action is transitive. Therefore, by the Orbit-Stabilizer Theorem, $[G:H]$ is divisible by the cardinality of Σ/H, i.e. by n. Consider any two H-orbits Σ_i and Σ_j, let $x_i \in \Sigma_i$ and $x_j \in \Sigma_j$, so that $\Sigma_i = Hx_i$ and $\Sigma_j = Hx_j$. By transitivity of G on Σ, there exists $g \in G$ such that $gx_i = x_j$. Then the map $\Sigma_i \to \Sigma_j$, $y \mapsto gy$ is a bijection. □

1.7.5 Generalized dihedral groups

We recall the following elementary fact.

Lemma 1.7.13. *Subgroups of the dihedral group $\mathrm{D}_n = \langle r,s \mid r^n = s^2 = (sr)^2 = \mathrm{id}\rangle$ are the following:*

Cyclic $\langle r^d\rangle \simeq \mathbb{Z}_{n/d}$, and $\langle r^k s\rangle$, where d divides n and $0 \leqslant k \leqslant n-1$.
Dihedral $\langle r^d, r^k s\rangle \simeq \mathrm{D}_{n/d}$, where $d < n$ divides n, and $0 \leqslant k \leqslant d-1$.

All cyclic subgroups $\langle r^d\rangle$ are normal, one has $\mathrm{D}_n/\langle r^d\rangle \simeq \mathrm{D}_d$, and these are all normal subgroups when n is odd. When n is even, there are two additional normal dihedral subgroups of index 2, namely $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$.

Let A be an abelian group. Recall that the *generalized dihedral group* $\mathrm{D}(A)$ is the semi-direct product $A \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by inverting the elements of A. In particular, for $A \simeq \mathbb{Z}_n$ one has $\mathrm{D}(\mathbb{Z}_n) \simeq \mathrm{D}_n$.

Lemma 1.7.14. *Let $n,m \geqslant 3$ be odd integers. Then one has the following:*

1. *Any group $G = \mathrm{D}_n \times_Q \mathrm{D}_m$ is either $\mathrm{D}_n \times \mathrm{D}_m$, or a generalized dihedral group $\mathrm{D}(A)$, where A is a direct product of at most two cyclic groups.*
2. *Furthermore, $G \simeq \mathrm{D}(A)$ admits a faithful 2-dimensional representation if and only if A is cyclic, so that G is isomorphic to a dihedral group.*

Proof. Fix the presentations $\mathrm{D}_n = \langle r_1, s_1 \mid r_1^n = s_1^2 = (s_1 r_1)^2 = \mathrm{id}\rangle$, $\mathrm{D}_m = \langle r_2, s_2 \mid r_2^m = s_2^2 = (s_2 r_2)^2 = \mathrm{id}\rangle$. We then have the following possibilities for G:

(i) $Q \simeq \mathrm{id}$. In this case, Lemma 1.7.9 implies that $G \simeq \mathrm{D}_n \times \mathrm{D}_m$.

(ii) $Q \simeq \mathbb{Z}_2$. Since n and m are odd, Lemma 1.7.9 and Lemma 1.7.13 show that G is an extension of $\mathbb{Z}_2$ by $A = \mathbb{Z}_n \times \mathbb{Z}_m$. Let $\tau \in G$ be an element mapped to the generator of Q. It necessarily belongs to $\mathrm{D}_n \times \mathrm{D}_m \setminus A$ and, being in the fibre product, is of the form $(s_1 r_1^k, s_2 r_2^l)$. The conjugation by τ induces an inversion on A, hence the claim.

(iii) $Q \simeq \mathrm{D}_q$, where q divides both n and m. Then the group G fits into the short exact sequence

$$1 \longrightarrow \mathbb{Z}_{n/q} \times \mathbb{Z}_{m/q} \longrightarrow G \xrightarrow{\psi} \mathrm{D}_q \longrightarrow 1,$$

where the groups $\mathbb{Z}_{n/q}$ and $\mathbb{Z}_{m/q}$ are generated by r_1^q and r_2^q, respectively. Let $A = \psi^{-1}(\mathbb{Z}_q)$. This is an index 2 subgroup in G of odd order; therefore, all its elements are of the form (r_1^k, r_2^l). Hence A is a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_m \subset \mathrm{D}_n \times \mathrm{D}_m$. As in the previous case, we conclude that $G \simeq \mathrm{D}(A)$.

To prove the second claim, it is enough to notice that $\mathrm{Z}(\mathrm{D}(A)) = \mathrm{id}$ for any non-trivial $A \subset \mathbb{Z}_n \times \mathbb{Z}_m$, since n and m are odd. In particular, if G admits a faithful 2-dimensional representation, then G can be embedded in $\mathrm{PGL}_2(\mathbb{C})$, hence A is a cyclic group. The converse is obvious. □

Chapter 2

G-solid rational surfaces

"Mathematics is not about waiting for the storm to pass, it's about learning how to dance in the rain."

Seneca, a math version – especially fitting in Edinburgh.

We classify G-solid rational surfaces over the field of complex numbers for finite-group actions. The results presented in this chapter have been published in *European Journal of Mathematics*, see Pinardin (2024).

2.1 Introduction

We are interested in the equivariant birational geometry of rational surfaces over the field of complex numbers for finite-group actions. Let S be a rational surface, and G be a finite group acting faithfully and biregularly on S. Let $\mathrm{rk}(\mathrm{Pic}^G(S))$ denote the rank of the G-invariant part of the Picard group of S. The G-equivariant Minimal Model Program applied to a resolution of singularities of S implies that S is G-birational to a *G-Mori fibre space*, that is, a G-surface in one of the following two cases:

- A G-del Pezzo surface, namely a del Pezzo surface S such that $-K_S$ is ample and $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$,
- A G-conic bundle, *i.e.* there is a G-equivariant morphism $S \to \mathbb{P}^1$ with general fibre isomorphic to $\mathbb{P}^1$, and such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 2$.

We say that S is *G-solid* if it is not G-birational to any G-conic bundle.

Main Theorem 2.1.1. *Let S be a G-del Pezzo surface of degree $d = K_S^2$. Then S is G-solid if and only if:*

- $d \leq 3$.
- $d = 4$ *and G does not fix a point on S in general position.*

- *$d = 5$ and G is not isomorphic to $\mathbb{Z}_5$ or D_5.*
- *$d = 6$ and G is not isomorphic to $\mathbb{Z}_6$, $\mathfrak{S}_3$, or D_6.*
- *$d = 8$, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, and, up to conjugation in $\mathrm{Aut}(S)$, either*
  - *G has a subgroup isomorphic to $\mathfrak{A}_4$,*
  - *or $G_4 \subset G$ and $G \not\subset G_{16}$, for two specific groups G_4 and G_{16}.*
- *$S \cong \mathbb{P}^2$, the group G does not fix a point on S, and is not isomorphic to $\mathfrak{S}_4$ or $\mathfrak{A}_4$.*

For $S = \mathbb{P}^2$, we also prove the following result.

Theorem 2.1.2. *Let G be a finite subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ isomorphic to $\mathfrak{A}_4$ or $\mathfrak{S}_4$. The only G-Mori fibre spaces G-birational to $\mathbb{P}^2$ are $\mathbb{P}^2$ and a del Pezzo surface of degree 5 with a G-conic bundle structure.*

In particular, for $G \cong \mathfrak{A}_4$ and for $G \cong \mathfrak{S}_4$, the projective plane is not G-birational to any Hirzebruch surface $\mathbb{F}_n$.

A recent motivation for our work is given in Tschinkel et al. (2023). Both G-solid surfaces and surfaces which are not G-birational to a Hirzebruch surface are classes of divisors on threefolds which give rise to *Incompressible Divisorial Symbols*, a modern tool used in the formalism of Burnside groups to distinguish birational types of group actions. An example of an application of these techniques by Cheltsov, Tshinkel, and Zhang can be found in Cheltsov, Tschinkel, and Zhang (2023b).

Finally, let us point out that the classification of G-solid Fano varieties is widely open starting from dimension 3. Some work has been achieved in the case of toric Fano threefolds, see, for example, Cheltsov, Dubouloz, and Kishimoto (2023), Cheltsov, Sarikyan, and Zhuang (2023), and Cheltsov, Tschinkel, and Zhang (2023b).

2.2 Del Pezzo surfaces of degree at most 5

Recall that by Manin–Segre's theorem 1.6.1, del Pezzo surfaces of degree one are birationally superrigid, and del Pezzo surfaces of degrees two and three are birationally rigid. For $K_S^2 = 4$, the result is essentially a corollary of the classification of G-links in Iskovskikh (1996).

Proposition 2.2.1. *Let G be a subgroup of $\mathrm{Aut}(S)$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, where S is a del Pezzo surface of degree 4. The following conditions are equivalent.*

- *S is G-rigid,*
- *S is G-solid,*
- *G does not fix a point on S outside of the (-1)-curves.*

Proof. Using the classification of Sarkisov links in Iskovskikh (1996), we see that there are three possible links starting from S. The first is the blow-up of a point not lying on a (-1) curve, leading to a G-conic bundle. The second is a Geiser involution, centered at an orbit of length 2, and the last is a Bertini involution, centered at an orbit of length 3. But as mentioned in Yasinsky (2023), Geiser and Bertini involutions lead to G-isomorphic surfaces. □

Starting from degree five, a deeper study of the group actions on del Pezzo surfaces is required. Recall that, up to isomorphism, there is only one smooth del Pezzo surface S of degree 5, given by the blow-up of $\mathbb{P}^2$ in four points in general position. Its group of automorphisms is isomorphic to $\mathfrak{S}_5$, and its description can be found in Blanc (2006).

Lemma 2.2.2 (Dolgachev and Iskovskikh (2009)). *Let $G \subset \mathrm{Aut}(S)$. Then $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$ if and only if G is isomorphic to $\mathfrak{S}_5$, $\mathfrak{A}_5$, D_5, $\mathbb{Z}_5$, or $\mathrm{F}_5 = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$*[1].

Recall the groups for which the G-solidity of S is known. The cases of $\mathfrak{A}_5$ and $\mathfrak{S}_5$ were solved by Cheltsov and $G = \mathrm{F}_5$ by Wolter.

Proposition 2.2.3 (Cheltsov (2008), Cheltsov (2014)). *If $G \subset \mathrm{Aut}(S)$ is isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$, then S is G-superrigid.*

Proposition 2.2.4 (Wolter (2018)). *If $G \subset \mathrm{Aut}(S)$ is isomorphic to F_5, then S is G-solid.*

What remains to study is the G-solidity of S for G isomorphic to $\mathbb{Z}_5$ or D_5. We will manage to avoid studying the G orbits on S, and we only need to use the G-birational geometry of $\mathbb{P}^2$.

Proposition 2.2.5. *If G is isomorphic to $\mathbb{Z}_5$ or D_5, then S is not G-solid.*

Proof. Consider the matrices $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_5 & 0 \\ 0 & 0 & \omega_5^{-1} \end{pmatrix} \in \mathrm{PGL}_3(\mathbb{C})$, where ω_5 is a primitive fifth root of unity, and $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. We have $H = \langle M \rangle \cong \mathbb{Z}_5$, and $H' = \langle M, N \rangle \cong \mathrm{D}_5$, under the action of both groups, the point $(1:1:1)$ has an orbit of length 5 in general position. Blowing up this orbit and contracting the proper transform of the conic passing through the five points gives a G-link from $\mathbb{P}^2$ to a del Pezzo surface of degree 5, for any $G \in \{H, H'\}$. On the other hand, for any $G \in \{H, H'\}$, the point $(1:0:0) \in \mathbb{P}^2$ is fixed under the action of G. Hence, we can blow it up and get a G-link to the Hirzebruch surface $\mathbb{F}_1$, with a G-conic bundle structure. Since $\mathbb{Z}_5$ and D_5 are unique in $\mathfrak{S}_5$ up to conjugacy, we conclude that S is not G-solid for any subgroup G of $\mathrm{Aut}(S)$ isomorphic to $\mathbb{Z}_5$ or to D_5. □

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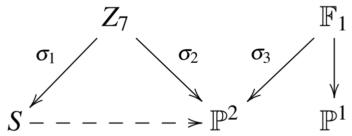
1. This is the group of GAP ID $(20,3)$.

2.3 Del Pezzo surfaces of degree 8

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. Recall that this is the only del Pezzo surface of degree 8 we have to study, since the blow-up of $\mathbb{P}^2$ at a point cannot be a G-del Pezzo surface. The automorphism group of S is isomorphic to $(\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on the direct product $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ by permuting its factors. We will use affine coordinates x and y and have the two $\mathrm{PGL}_2(\mathbb{C})$ components act on them, respectively. The action of $\mathbb{Z}_2$ is given by the permutation of x and y. For example, an automorphism written $(x_0 : x_1) \times (y_0 : y_1) \mapsto (y_1 : y_0) \times (x_0 : x_1)$ will be denoted as $(\frac{1}{y}, x)$. To begin with, we will give an example of a group $G \subset \mathrm{Aut}(S)$ such that the surface S is not G-solid.

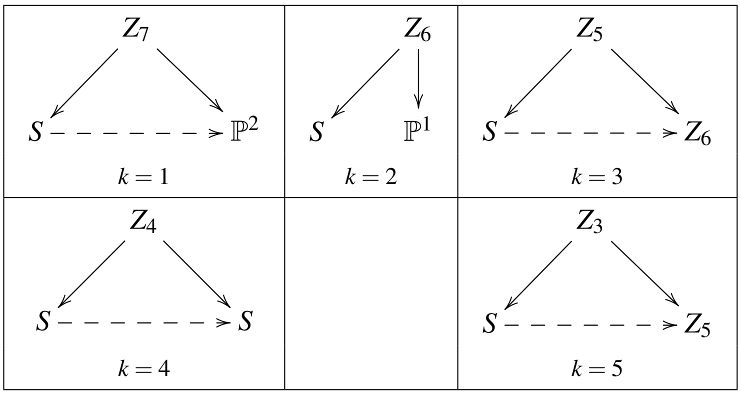
Example 2.3.1. Let π_1 and π_2 be the two canonical projections from S to $\mathbb{P}^1$. If two points P and Q are such that $\pi_i(P) \neq \pi_i(Q)$ for $i = 1$ and $i = 2$, we say that they are *in general position*. Assume that a subgroup $G \subset \mathrm{Aut}(S)$ fixes two points in general position. Then there is a G-birational map from S to the G-conic bundle $\mathbb{F}_1$, which decomposes into two G-links as follows:



where σ_1, σ_2 and σ_3 are blow-ups at a point. Here is a list of explicit cases in which it occurs:

- If $G = \langle s, t_n \rangle$, with $s = (y, x)$, and $t = (\omega_n x, \omega_n^{-1} y)$, then $G \cong \mathrm{D}_n$.
- If $G = \langle \sigma, \tau_n \rangle$, with $\sigma = (y, -x)$ and $\tau_n = (ix, iy)$, then $G \cong Q_8$.

Remark 2.3.2. Let $\varphi\colon S \dashrightarrow S'$ be a G-link centred at k points. Using the classification of Sarkisov links in Iskovskikh (1996), we see that, if $k \geq 6$, then φ is either a Bertini or a Geiser involution. Once again, as mentioned in Yasinsky (2023), such link leads to a G-isomorphic surface. It follows that if S' is not G-isomorphic to S, then φ is of one of the following forms.



A surface denoted by Z_k is a del Pezzo surface of degree k. We deduce that if G does not have any orbit of length $k \leq 5$, then S is G-rigid. One can mention that every pair of points in the centre of one of the links of Remark 2.3.2 must be in general position, as defined in Example 2.3.1.

2.3.1 Toric subgroups of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$

We can embed $(\mathbb{C}^*)^2$ as a dense torus in S by the map $\iota\colon (\mathbb{C}^*)^2 \hookrightarrow S$, $(a,b) \mapsto (1:a)\times(1:b)$, whose image will be called $\mathfrak{T}$. Moreover, any dense torus in S is equal to $\mathfrak{T}$ up to an automorphism of S. The action of $(\mathbb{C}^*)^2$ on itself by translation extends to a faithful action on the whole variety S, identifying $(\mathbb{C}^*)^2$ with the subgroup $\mathbb{T} = \{(ax,by), a,b \in \mathbb{C}^*\}$ of $\mathrm{Aut}(S)$.

Lemma 2.3.3. *There is an exact sequence*

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T}) \xrightarrow{w} \mathrm{D}_4 \longrightarrow 1.$$

Proof. First, notice that $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ leaves $\mathfrak{T}$ invariant. Indeed, the normalizer of $\mathbb{T}$ in $\mathrm{Aut}(S)$ permutes the $\mathbb{T}$-orbits, and $\mathfrak{T}$ is the only one that is dense in S. The complement of $\mathfrak{T}$ in S is the divisor $C = \pi_1^{-1}(1:0) + \pi_1^{-1}(0:1) + \pi_2^{-1}(1:0) + \pi_2^{-1}(0:1)$, where π_1 (*resp.* π_2) is the canonical projection from $\mathbb{P}^1 \times \mathbb{P}^1$ to the left (*resp.* right) factor $\mathbb{P}^1$. The intersection number being preserved by automorphisms, any element of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ induces a symmetry of the square formed by C, thus giving a group homomorphism $w\colon \mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T}) \to \mathrm{D}_4$. Finally, the group $\mathbb{T}$ is the set of automorphisms that preserve each irreducible curve of the divisor C. In other words, the kernel of w is $\mathbb{T}$. Moreover, w is surjective, hence the exact sequence. □

Remark 2.3.4. Notice that $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ is exactly the set of automorphisms that preserve the square C. This will be useful later to prove that a subgroup G of $\mathrm{Aut}(S)$ is contained in $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$.

Let G be a subgroup of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$, and $T = G \cap \mathbb{T}$. The restriction of w to G induces the exact sequence

$$1 \longrightarrow T \longrightarrow G \xrightarrow{w} W \longrightarrow 1,$$

for some subgroup W of D_4.

We will call *toric* a subgroup of $\mathrm{Aut}(S)$ conjugated to a subgroup of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$. Consider the automorphisms $r = (\frac{1}{y}, x)$, and $s = (y,x)$. Both belong to $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$, and they generate a group isomorphic to D_4, on which w restricts to an isomorphism. In Cheltsov, Dubouloz, and Kishimoto (2023), the authors mention without proof that if G is a subgroup of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, then its image W in D_4 must contain $\mathbb{Z}_4$. We prove this result here and state it in a slightly stronger way, adding that an element mapped onto $w(r)$, thus generating $\mathbb{Z}_4$ in D_4, is equal to r up to conjugation by an element of the torus.

Lemma 2.3.5. *Let G be a toric subgroup of* $\mathrm{Aut}(S)$ *such that* $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$*. If S is G-solid, then G is conjugated by an element of* $\mathbb{T}$ *to a subgroup of* $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ *containing r.*

Proof. Since G is toric, we may assume that it is a subgroup of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$. The orbits of the vertices of the square C under the action of D_4 are of length 1, 2, or 4. If one of them is fixed, the opposite vertex must be fixed as well, and we get the situation of Example 2.3.1. In particular, the surface S is not G-solid. If no vertex is fixed, then the vertices are on the same orbit or form two orbits of length 2. Since $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, we cannot have two orbits of length 2, each of them consisting of two consecutive vertices of C. So these two orbits must be formed by opposite vertices, and we deduce that there are two orbits of length 2 in general position. We can blow up one of them to get a G-link to a G-conic bundle. Hence, the only possibility for S to be G-solid is that all the vertices are on the same orbit. In this case, since $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, the vertices must be cyclically permuted, so $w(r) \in W$. It remains to note that if an element $g \in \mathrm{Aut}(S)$ satisfies $w(g) = w(r)$, then g is conjugated to r in $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$. Indeed, such automorphism g is of the form $(\frac{a}{y}, bx)$, for some $a, b \in \mathbb{C}^*$. Let $t = (kx, ly) \in \mathbb{T}$, with $k, l \in \mathbb{C}^*$. If $l^2 = \frac{1}{ab}$ and $k = bl$, then we have $tgt^{-1} = r$. ☐

Lemma 2.3.6. *Let G be a subgroup of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ containing r and $T = G \cap \mathbb{T}$. If $|T| > 5$, then S is G-rigid.*

Proof. The G-orbits of the points outside of the square C are of length at least T. According to Remark 2.3.2, there is no link centered at such an orbit leading to a non-G-isomorphic surface. Moreover, the orbit of a point lying in the square C is not in general position. ☐

We will now begin the exhaustive study of the subgroups G of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, where $|G \cap \mathbb{T}| \leq 5$ and $r \in \mathbb{T}$. Let G be such a subgroup of $\mathrm{Aut}(S)$. First, we can eliminate the cases where $G \cong \mathbb{Z}_3$ or $G \cong \mathbb{Z}_4$.

Lemma 2.3.7. *Let G be a subgroup of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ containing r. Then the toric part $T = G \cap \mathbb{T}$ cannot be isomorphic to $\mathbb{Z}_3$ or to $\mathbb{Z}_4$.*

Proof. Assume that T is isomorphic to $\mathbb{Z}_3$ (*resp.* $\mathbb{Z}_4$). Then T is generated by an automorphism $t \in \mathbb{T}$ of the form $(\omega_n x, \omega_n^k y)$ or $(\omega_n^k x, \omega_n y)$, where $n = 3$ (*resp.* 4), and ω_n is a primitive n-th root of the unity. But by Lemma 2.4.1, we may assume that G contains the element $r = (\frac{1}{y}, x)$, acting on T by conjugation. Since $(r(\omega_n^k x, \omega_n y)r^{-1})^{-1} = (\omega_n x, \omega_n^{-k} y)$, we may assume that t is of the first form, that is, we have $T = \langle(\omega_n x, \omega_n^k y)\rangle$. The automorphism $rtr^{-1} = (\omega_n^{-k} x, \omega_n y) \in T$ must belong to T, and therefore must be a power $(\omega_n x, \omega_n^k y)$. It is impossible for $n = 3$ or $n = 4$. ☐

Proposition 2.3.8. *If T is trivial, then S is not G-solid. The options for G are the following.*

- *A group isomorphic to $\mathbb{Z}_4$, generated by $r = (\frac{1}{y}, x)$,*
- *A group isomorphic to D_4, generated by r and $s = (y, x)$,*
- *A group isomorphic to D_4, generated by r and $(-y, -x)$.*

Proof. The toric part T is trivial, so the map w restricts to an isomorphism on G. Since $r \in G$, we either have $G \cong \mathbb{Z}_4$, or $G \cong \mathrm{D}_4$. Assume we are in the latter case. Then G is generated by r, and an element h such that $w(h) = w(s)$, or in other words such that $h = ts$, for some $t = (ax, by) \in \mathbb{T} \cong (\mathbb{C}^*)^2$. Since w restricts to an isomorphism on G, we can write $hrh = w^{-1}(srs) = w^{-1}(r^{-1}) = r^{-1}$. It yields $r^{-1} = (a^2 y, \frac{1}{x})$, so that $a^2 = 1$. Finally, since h is of order

2, we get $ab = 1$. So the only two options for h are $h = s$, or $h = (-y, -x)$. If G is generated by r and s, the points $(1,1)$ and $(-1,-1)$ are fixed by the action, in which case S is not G-solid, as in example 2.3.1. If G is generated by r and $(-y,-x)$, the points $(1,1)$ and $(-1,-1)$ form an orbit of length 2. We can G-equivariantly blow-up these points and obtain a G-link to a G-conic bundle. Finally, if $\mathbb{Z}_4 \cong G = \langle r \rangle$, then G is a subgroup of $\langle r, s \rangle$, so that the surface S is not G-solid. □

Proposition 2.3.9. *If $T \cong \mathbb{Z}_2$, there are the following possibilities and only them:*

- $G = \langle t, r \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, *with $t = (-x, -y)$. The surface S is not G-solid.*
- $G = \langle t, r, s \rangle \cong \mathbb{Z}_2 \times \mathrm{D}_4$. *The surface S is not G-solid.*
- $G = \langle t, r, h \rangle \cong \mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$, *with t and g as above, and $h = (-y, x)$.The surface S is G-solid.*

Proof. Assume that $w(G) = \mathrm{D}_4$. Then G is generated by r, an element $t \in T$, and an element h such that $w(h) = (ay, bx)$, for some $a, b \in \mathbb{C}^*$. Since T is of order 2, the element t is $(-x,-y), (-x,y)$ or $(x,-y)$. But the relation $rtr^{-1} = t$ implies that the only possibility is $t = (-x,-y)$. The order of h can only be 2 or 4. If $\mathrm{Ord}(h) = 2$, then $a = b^{-1}$. Since $w(hrh) = w(r^{-1})$, we have $hrh = \tau r^{-1} = \tau \cdot (y, \frac{1}{x})$, for some $\tau \in T$, *i.e.*, $\tau = \mathrm{id}$ or $\tau = t$. But $hrh = \tau \cdot (a^2 y, \frac{1}{x})$, which implies $a = b = \pm 1$. But since $t \in G$, we may assume that $a = b = 1$. The points $P_1 = (1:1) \times (1:1)$ and $P_2 = (1:-1) \times (1:-1)$ form an orbit of length 2, and blowing it up gives a G-link to a G-conic bundle.

Assume $\mathrm{Ord}(h) = 4$. Since $h^2 = (abx, aby)$, this implies that $a^2 b^2 = 1$ and $ab \neq 1$, so that $ab = -1$ and $h = (ax, -\frac{y}{a})$. Using again the fact that $r^{-1} = \tau h r h^{-1} = \tau \cdot (a^2 y, \frac{1}{x})$, for some $\tau \in T$, we deduce that $h = (-y, x)$ or $h = (y, -x)$. Since these possibilities only differ by t, we may assume that $h = (-y, x)$. The group G is generated by t, r, and h, and is isomorphic to $G \cong \mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$. There is no G-orbit of length $l \leq 5$, therefore S is G-solid.

Finally, if $w(G) = \mathbb{Z}_4$, then $\mathbb{Z}_2 \times \mathbb{Z}_4 \cong G = \langle t, r \rangle \subset \langle t, r, s \rangle$, so that the surface S is not G-solid, as in the case where $G = \langle t, r, s \rangle$. □

Proposition 2.3.10. *If $T \cong \mathbb{Z}_2^2$, then S is G-solid. Moreover, there are the following possibilities for G, and only them:*

- $G \cong \mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$, *generated by r and $t = (-x, y)$,*
- $G \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$, *generated by r, $t = (-x, y)$, and $s = (y, x)$,*[2]
- $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$, *generated by r, $t = (-x, y)$, and $h = (iy, ix)$.*[3]

---

2. This semidirect product is the group of GAP ID $(32, 27)$.
3. This semidirect product is the group of GAP ID $(32, 6)$.

Proof. Assume that $w(G) \cong \mathbb{Z}_4$, so that $G = \langle T, r\rangle$. Since $T \subset (\mathbb{C}^*)^2$ is isomorphic to $\mathbb{Z}_2^2$, it is generated by $t_1 = (-x, y)$ and $t_2 = (x, -y)$. The group G is isomorphic to $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$. Notice that since $rt_1r^{-1} = t_2$, the group G is generated by r and t_1. There is no G-orbit of length $l \leq 5$. Hence, the surface S is G-solid. If $w(G) = \mathrm{D}_4$, the group G contains $\langle r, t_1\rangle$. Hence, the surface S is also G-solid.

We now complete the list of groups whose toric part is isomorphic to $\mathbb{Z}_2^2$. Assume that $w(G) = \mathrm{D}_4$, so that there exists an element $h \in G$ such that $w(h) = w(s)$. The order of h is 2 or 4. If $\mathrm{Ord}(h) = 2$, we get $h = (ay, \frac{x}{a})$. But $r^{-1} = \tau hrh^{-1} = \tau \cdot (a^2y, \frac{1}{x})$, so $\tau = \mathrm{id}$ and $a = \pm 1$, or $\tau = t_1 = (-x, y)$ and $a = \pm i$. Up to composition by an element of T, we get $h = s = (y, x)$ or $h = (-ix, iy)$. In the first case, we have that $G = \langle t_1, t, h\rangle$ is isomorphic to a semidirect product of the form $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$, and in the second case to a semidirect product of the form $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$. If $\mathrm{Ord}(h) = 4$, we get $h = (ay, -\frac{x}{a})$, which only differs from the previous case by an element of T, so G is still $\langle t_1, r, s\rangle$, or $\langle t_1, r, (-ix, iy)\rangle$. □

The only remaining possibility is $T \cong \mathbb{Z}_5$. We will use the results in Wolter (2018) on the G-solidity of the del Pezzo surface of degree 5.

Proposition 2.3.11. *If $T \cong \mathbb{Z}_5$, then G is isomorphic to the Fröbenius group $\mathrm{F}_5 \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$, generated by r and $(\omega_5 x, \omega_5^l y)$ for $l = 2$ or $l = 3$, where ω_5 is a primitive fifth root of the unity.*

Proof. Assume $T \cong \mathbb{Z}_5$. It is generated by an element t of the form $t = (\omega_5 x, \omega_5^l y)$ or $t = (\omega_5^k x, \omega_5 y)$. Since $r(\omega_5^k x, \omega_5 y)^{-1} r^{-1} = (\omega_5 x, \omega_5^{-k} y)$, we may assume that $t = (\omega_5 x, \omega_5^l y)$. But then $rtr^{-1} = (\omega_5 x^{-l}, \omega_5 y)$, and this element is in $\langle t\rangle$ if and only if $l = 2$ or $l = 3$. Assume that there is an element $h \in G$ such that $w(h) = w(s)$. Then it is of the form (ay, bx), for some $a, b \in \mathbb{C}^*$. But $hth^{-1} = (\omega_5^l x, \omega_5 y)$, which is not in T. Since this subgroup is normal in G, we get a contradiction. Hence, G is generated by $(\omega_5 x, \omega_5^l y)$ and r, with $l = 2$ or $l = 3$. In both cases, G is isomorphic to the Fröbenius group $\mathrm{F}_5 \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$. The points outside of C have orbits of length at least 5, so the only possible G-Sarkisov that does not lead to a G-isomorphic surface leads to a del Pezzo surface of degree 5 with invariant Picard rank 1. This surface is G-solid, according to Wolter (2018). □

2.3.2 Non-toric subgroups of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$

Proposition 2.3.12. *Let G be a subgroup of $\mathrm{Aut}(S)$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Up to conjugation of G in $\mathrm{Aut}(S)$, there is an exact sequence of the form*

$$1 \longrightarrow H \times_D H \longrightarrow G \xrightarrow{\ \delta\ } \mathbb{Z}_2 \longrightarrow 1.$$

where H is the projection of G onto the first and second factor $\mathrm{PGL}_2(\mathbb{C})$ in $(\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) \rtimes \mathbb{Z}_2$.

Before proving the above result, recall that the *fibre product* of two groups H and H', over a group D, for the surjective group morphisms $\varphi\colon H \to D$ and $\psi\colon H' \to D$, is the subgroup $\{(h,h'), \varphi(h) = \psi(h')\}$ of $H \times H'$. In some of our proofs, we will have to be particularly careful about the surjective group morphisms φ and ψ, in which case we will use the notation $H \times_{D,\varphi,\psi} H$, instead of $H \times_D H$.

Proof of Proposition 2.3.12. Define the group morphism $\delta\colon \mathrm{Aut}(S) \to \mathbb{Z}_2$ that sends an element g to 1 if and only if g swaps the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. Since $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, there must be such an element in G, hence the surjectivity of the restriction of δ to G. The kernel of δ in $\mathrm{Aut}(S)$ is $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$, and therefore the kernel of the restriction of δ to G is a subgroup of $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$. Applying Lemma 1.7.8, it is of the form $H \times_D H'$, where H and H' are subgroups of $\mathrm{PGL}_2(\mathbb{C})$. For the remainder of this proof, we will denote elements of $\mathrm{Aut}(S) \cong (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) \rtimes \mathbb{Z}_2$ as triples (h,h',a), where $h,h' \in \mathrm{PGL}_2(\mathbb{C})$, and $a \in \mathbb{Z}_2$. Let $(h,h',0) \in \ker \delta_G$, and let $g = (a,a',1) \in G$. Then $g(h,h',0)g^{-1} = (ah'a^{-1}, a'ha'^{-1}, 0)$. Since $\ker \delta_G$ is normal in G, it yields $aH'a^{-1} = H$. Let $\alpha = (a,I,0)$, and denote $\varphi\colon H \to D$, $\psi\colon H' \to D$ the morphisms of the fibre product. We have $\alpha(H \times_{D,\varphi,\psi} H')\alpha^{-1} = H \times_{D,\varphi,\xi} H$, where

$$
\begin{aligned}
\xi\colon H &\to D \\
h &\mapsto \psi(a^{-1}ha),
\end{aligned}
$$

and we have $\ker(\delta_{\alpha G \alpha^{-1}}) = (H \times_{D,\varphi,\xi} H) \times \{e\}$. □

Let G be a finite subgroup of $\mathrm{Aut}(S)$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, and H as in Proposition 2.3.12.

Lemma 2.3.13. *If G is not toric, then H is isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$.*

Proof. Assume that H is isomorphic to $\mathbb{Z}_n$ or D_n. Then there are points P_1 and P_2 of $\mathbb{P}^1$ that are fixed by H or that form an orbit of length 2. Consider the divisors $L_1 = \{(P_1,y), y \in \mathbb{P}^1\}$, and $L_2 = \{(P_2,y), y \in \mathbb{P}^1\}$. The divisor $L_1 + L_2$ is invariant by $H \times_D H$. Let $g \in G$ be such that G is generated by $H \times_D H$ and g. This element swaps the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, so that the divisor $D = L_1 + L_2 + g(L_1) + g(L_2)$ forms a square. Let $\alpha \in G$. Since $H \times_D H$ is normal in G and $g^2 \in H \times_D H$, we can write $\alpha = hg^i = g^ih'$, for some $h,h' \in H \times_D H$ and $i \in \{0,1\}$. We get:

$$
\begin{aligned}
\alpha(D) &= g^ih(L_1 + L_2) + g^ihg(L_1 + L_2) \\
&= g^i(L_1) + g^i(L_2) + g^{i+1}h'(L_1 + L_2) \\
&= g^i(L_1) + g^i(L_2) + g^{i+1}(L_1) + g^{i+1}(L_2) \\
&= D.
\end{aligned}
$$

Finally, the groups that fix this square are conjugated in $\mathrm{Aut}(S)$ to a subgroup of $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$. Indeed, there is an element $\xi \in \mathrm{PGL}_2(\mathbb{C})$ sending P_1 to $(1:0)$ and P_2 to $(0:1)$. Let $g = (\xi,\xi,0) \in (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) \times \mathbb{Z}_2 \cong \mathrm{Aut}(S)$. The image of D by g is the square C as in the proof of Lemma 2.3.3. Moreover, gGg^{-1} fixes C, so that gGg^{-1} is in $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$, as implied by Remark 2.3.4. □

Proposition 2.3.14. *If G is not toric, then there are no G-orbits of length $l \in \{1,2,3,5\}$.*

Proof. Let $P_1, P_2 \in \mathbb{P}^1$, and k_1, k_2 be the respective lengths of their H-orbits in $\mathbb{P}^1$. The length of the orbit of (P_1,P_2) in S under the action of $H \times_D H$ must be a common multiple l of k_1 and k_2, and the length of its G-orbit is either l or $2l$. Hence, by Lemma 2.3.13, it is enough to show that there is no $\mathfrak{A}_4$-orbit of length $l \in \{1,2,3,5\}$ in $\mathbb{P}^1$.

Assume that $H \simeq \mathfrak{A}_4$. Recall that $\mathfrak{A}_4$ is unique up to conjugation in $\mathrm{PGL}_2(\mathbb{C})$. A simple way[4] to see what the $\mathfrak{A}_4$-orbits in $\mathbb{P}^1$ are is to use the action of $\mathfrak{A}_4$ in $\mathrm{PGL}_3(\mathbb{C})$ generated by the matrices $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It preserves the conic $x^2+y^2+z^2=0$, hence acts on $\mathbb{P}^1$ faithfully. We see that there is no fixed point and that there is no orbit of length 2 since $\mathfrak{A}_4$ does not have any subgroup of index 2. The only subgroup of index 3 is $\mathbb{Z}_2^2$, generated by the matrices $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. But it has no fixed point on the conic, hence there is no H-orbit of length 3. Considering the order of $\mathfrak{A}_4$, there is no H-orbit of length 5.

If H is isomorphic to $\mathfrak{S}_4$ or $\mathfrak{A}_5$, and since $\mathfrak{A}_4, \mathfrak{S}_4$, and $\mathfrak{A}_5$ are unique in $\mathrm{PGL}_2(\mathbb{C})$ up to conjugation, the lengths H-orbits in $\mathbb{P}^1$ must be multiples of the lengths of the $\mathfrak{A}_4$-orbits, namely a multiple of $k \notin \{1,2,3,5\}$. This implies that there is no H-orbit of length $k \in \{1,2,3,5\}$ in $\mathbb{P}^1$. Summing up what we mentioned in this proof, a G-orbit can only be a multiple of k, for some positive integer $k \notin \{1,2,3,5\}$. □

Remark 2.3.15. There is a non-toric subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{A}_4 \rtimes \mathbb{Z}_2$, which has a G-orbit of length four in general position. Recall from Remark 2.3.2, that there is a G-link centered in such an orbit. It is of the following form.

$$\begin{array}{ccc} & Z_4 & \\ \swarrow & & \searrow \\ (S,G) & \dashrightarrow & (S,G') \end{array}$$

4. One could also generate $\mathfrak{A}_4$ explicitly in $\mathrm{PGL}_2(\mathbb{C})$, for example with the matrices $\begin{pmatrix} 1 & 0 \\ 0 & \omega_3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, where ω_3 is a primitive cube root of the unity.

By Proposition 2.3.14, the only way for S not to be G-solid, if G is a toric group with $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, would be for G' in the above diagram to be non-toric. To exclude this possibility, one could use a result of Yasinsky (2023) stating that such link must lead to a G-isomorphic surface. We will use another method based on general group theory. It will have the advantage of giving a simple characterization of the toric subgroups of $\mathrm{Aut}(S)$, making the final result much easier to read. It turns out that a finite toric subgroup G of $\mathrm{Aut}(S)$ with $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$ cannot be isomorphic to a non-toric subgroup of $\mathrm{Aut}(S)$ with $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$.

Proposition 2.3.16. *Let G be a subgroup of $\mathrm{Aut}(S)$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Then G is toric if and only if it has no subgroup isomorphic to $\mathfrak{A}_4$.*

We will step back to some general group theory to achieve the above result. To prove that a non-toric group contains $\mathfrak{A}_4$, it suffices to show that $H \times_D H$ contains a subgroup isomorphic to H. The fact that a fibre product of this form contains a subgroup isomorphic to H is not always true if we consider any group H, although many examples seem to show the contrary. In the following example, the group H has the smallest possible order such that there are surjective morphisms $\varphi, \psi \colon H \to D$ making the $H \times_D H$ a group that does not contain any subgroup isomorphic to H.

Example 2.3.17. Let $H \simeq \mathrm{D}_4 = \langle a, b | a^4 = b^2 = (ab)^2 = 1 \rangle$, take $D = \mathbb{Z}_2^2$, and

$$\begin{aligned} \varphi \colon a &\mapsto (0,1) \\ b &\mapsto (1,0) \\ \psi \colon a &\mapsto (1,0) \\ b &\mapsto (1,1). \end{aligned}$$

Then $H \times_D H$ is isomorphic to $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$, and does not have any subgroup isomorphic to H. This example exists in our context. Let $H = \langle a, b \rangle$, where $a = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and define the morphisms φ and ψ as above. To have a subgroup G of $\mathrm{Aut}(S)$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$ and $G \cap (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) = H \times_{\mathbb{Z}_2^2} H$, we take $G = \left\langle H \times_{\mathbb{Z}_2^2} H, g \right\rangle$, where $g = (\omega_8^{-1} y, \omega_8 x)$, with ω_8 a square root of i.

However, it is true that if a group H splits in a nice way, then a fibre product of the form $H \times_D H$ will have a subgroup isomorphic to H. It will apply in particular for the groups we need, namely $\mathfrak{A}_4, \mathfrak{S}_4$, and $\mathfrak{A}_5$.

Lemma 2.3.18. *Let H be a group of the form $N \rtimes D$, and $\varphi, \psi \colon H \to D$ be surjective group morphisms such that $\ker(\varphi) = \ker(\psi) = N$. Then $H \times_D H$ has a subgroup isomorphic to H.*

Proof. The subgroups $N = \ker(\psi)$ and D having a trivial intersection in H means that the restriction of ψ to D is an isomorphism. So we can define the set $\widetilde{D} = \{(d, \psi^{-1} \circ \varphi(d)), d \in D\}$. It is a subgroup of $H \times_D H$, isomorphic to D. Denote $\widetilde{N}$ the subgroup $\ker(\varphi) \times \{\mathrm{id}\}$ of $H \times H$. It is a subgroup of $H \times_D H$, isomorphic to $\ker(\varphi) = N$. Consider the projection π

from $H \times H$ to the first factor H. Its restriction to $\widetilde{N}\widetilde{D}$ is surjective by construction of $\widetilde{N}$ and $\widetilde{D}$. If $\pi(g,g') = e$, then $g = e$. But since $\widetilde{N}$ and $\widetilde{D}$ have a trivial intersection, $e = e.e$ is the only decomposition of e into a product of an element of N and an element of D. Hence, we get $(g,g') = (e.e, e.\psi^{-1} \circ \varphi(e)) = (e.e)$, so π restricts itself to an isomorphism between $\widetilde{N}\widetilde{D} \subset H \times_D H$ and H. $\square$

We get the following immediate consequence.

Corollary 2.3.19. *In the following cases, a fibre product of the form $H \times_D H$ contains a subgroup isomorphic to H.*

- *H is simple,*
- *H is cyclic,*
- *H is isomorphic to $\mathfrak{A}_n$ or $\mathfrak{S}_n$ for some n.*

We can now prove Proposition 2.3.16.

Proof of Proposition 2.3.16. Recall that the normalizer $\mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T})$ of the torus $\mathbb{T}$ in $\mathrm{Aut}(S)$ satisfies this exact sequence:

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{N}_{\mathrm{Aut}(S)}(\mathbb{T}) \xrightarrow{w} \mathrm{D}_4 \longrightarrow 1.$$

But there is no subgroup of D_4 isomorphic to a quotient of $\mathfrak{A}_4$ by an abelian group. Hence, if G is toric, it cannot have a subgroup isomorphic to $\mathfrak{A}_4$. Conversely, assume that G is a finite toric subgroup of $\mathrm{Aut}(S)$ with $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. By Proposition 2.3.12, it satisfies the exact sequence

$$1 \longrightarrow H \times_D H \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$

with H isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$. By Lemma 2.3.18, the group G has a subgroup isomorphic to H. Hence, it has a subgroup isomorphic to $\mathfrak{A}_4$. $\square$

Summing up the above results, we get the following.

Proposition 2.3.20. *Let G be a finite non-toric subgroup of $\mathrm{Aut}(S)$. Then S is G-solid.*

Finally, here is the classification of subgroups G of $\mathrm{Aut}(S)$ such that S is G-solid.

Theorem 2.3.21. *Let G be a finite subgroup of $\mathrm{Aut}(S)$, such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Then S is not G-solid if and only if G is toric, and in one of the following cases.*

- *The group G is not conjugated in $\mathrm{Aut}(S)$ to a group containing $r = (\frac{1}{y}, x)$.*
- *G is conjugated in $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ to one of the following groups.*
  - *$\mathbb{Z}_4$, generated by $r = (\frac{1}{y}, x)$,*
  - *D_4, generated by r, and (y,x),*
  - *D_4, generated by r and $(-y,-x)$,*
  - *$\mathbb{Z}_4 \times \mathbb{Z}_2$, generated by r and $t = (-x,-y)$,*

– $\mathbb{Z}_2 \times \mathrm{D}_4$, *generated by* t *and* r *as above, and* $h = (y,x)$.

Once again, Proposition 2.3.16 not only allows us to conclude about the G-solidity of the non-toric finite subgroups of $\mathrm{Aut}(S)$, but also gives a way to reformulate Theorem 2.3.21 without mentioning the toric structure of S, giving an equivalent statement which is shorter and easier to read. Let $G_{16} \cong \mathbb{Z}_2 \times \mathrm{D}_4$ be the subgroup of $\mathrm{Aut}(S)$ generated by $r = (\frac{1}{y}, x), s = (y,x)$, and $t = (-x,-y)$.

Theorem 2.3.22. *Let G be a finite subgroup of $\mathrm{Aut}(S)$, such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Then S is G-solid if and only if, up to conjugation in $\mathrm{Aut}(S)$,*

- *either $\mathfrak{A}_4 \subset G$,*
- *or ($r \in G$ and $G \not\subset G_{16}$).*

2.4 Del Pezzo surfaces of degree 6

Up to isomorphism, there is only one smooth del Pezzo surface of degree 6. It is obtained by blowing up $\mathbb{P}^2$ in three points P_1, P_2 and P_3 in general position. We will denote by E_i the exceptional curve contracted to the point P_i, and by D_{ij} the proper transform of the line passing through P_i and P_j. Recall that there is a split exact sequence

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Aut}(S) \xrightarrow{w} \mathrm{D}_6 \longrightarrow 1,$$

where w is given by the action of $\mathrm{Aut}(S)$ on the hexagon formed by the (-1)-curves of S. The group $\mathbb{T} \cong (\mathbb{C}^*)^2$ is the lift in $\mathrm{Aut}(S)$ of the diagonal automorphisms of $\mathbb{P}^2$, which fixes P_1, P_2, and P_3. This subgroup will be denoted by $\mathbb{T}$.

We will use the embedding of S in $\mathbb{P}^2 \times \mathbb{P}^2$ given by $xu = yv = zw$, where $(x:y:z) \times (u:v:w)$ stands for the coordinates in $\mathbb{P}^2 \times \mathbb{P}^2$. This model is presented in more detail in Blanc (2006). Explicitly, an element $(a,b) \in (\mathbb{C}^*)^2$ corresponds to the map $(x:y:z) \times (u:v:w) \to (x:ay:bz) \times (u:a^{-1}v:b^{-1}w)$. The maps

$$r\colon (x:y:z) \times (u:v:w) \mapsto (w:u:v) \times (z:x:y), \text{ and}$$
$$s\colon (x:y:z) \times (u:v:w) \mapsto (x:z:y) \times (u:w:v)$$

generate a subgroup of $\mathrm{Aut}(S)$ isomorphic to D_6, and the quotient $\mathrm{Aut}(S)/\mathbb{T} \cong \mathrm{D}_6$ is generated by the images of r and s. The automorphism r acts on the hexagon formed by the (-1)-curves as an elementary rotation, and s acts as a reflection of the hexagon which does not fix any vertex. We have the relations $r^6 = s^2 = (rs)^2 = \mathrm{id}$, giving the classical presentation of D_6.

Lemma 2.4.1. *Any element $r' \in \mathrm{Aut}(S)$ such that $w(r') = w(r)$ is equal to r up to conjugation by an element of $\mathbb{T}$.*

Proof. Let $r' \in \mathrm{Aut}(S)$ be such that $w(r') = w(r)$. Since the kernel of w is the normal subgroup $\mathbb{T}$ of $\mathrm{Aut}(S)$, we have $r' = tr$, for some $t \in \mathbb{T}$. Explicitly, there exist $(a,b) \in (\mathbb{C}^*)^2$ such that $r' \colon (x:y:z) \times (u:v:w) \mapsto (w:au:bv) \times (z:a^{-1}x:b^{-1}y)$. Let $t \colon (x:y:z) \times (u:v:w) \mapsto (x:cy:dz) \times (u:c^{-1}v:d^{-1}w) \in \mathbb{T} \cong (\mathbb{C}^*)^2$. We have

$$tr't^{-1} \colon (x:y:z) \times (u:v:w) \mapsto (w:acd^{-1}u:bcv) \times (z:a^{-1}c^{-1}dx:b^{-1}c^{-1}y).$$

Setting $c = b^{-1}$ and $d = ab^{-1}$, we get $tr't^{-1} = r$. ☐

Let G be a subgroup of $\mathrm{Aut}(S)$, such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. The classification of Sarksov links in Iskovskikh (1996) and the fact that Bertini and Geiser involutions lead to G-isomorphic surfaces imply the following.

Remark 2.4.2. There is no link of type I starting from S. Hence, for S not to be G-solid, it has to be G-birational to a surface S', not isomorphic to S. The only G-link $S \dashrightarrow S'$ such that S' is not isomorphic to S is the blow-up of a point $P \in S$ that is not in the exceptional locus, followed by the contraction of three (-1)-curves. We obtain $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Hence, any G-birational map from S to a G-conic bundle $S'' \to \mathbb{P}^1$ must split in the following way:

$$S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow S''.$$

We will often refer to the results of section 2.3 to determine whether or not the surface S is G-solid.

In particular, Remark 2.4.2 implies the following lemma:

Lemma 2.4.3. *Let G be a subgroup of $\mathrm{Aut}(S)$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, not isomorphic to a subgroup of D_6. Then S is G-solid.*

Proof. Assume that S is not G-solid. Then, by Remark 2.4.2, there exists a subgroup G' of $\mathrm{Aut}(S)$ birationally conjugated to G, which fixes a point P in general position. But in this case, $G' \cap \mathbb{T} = \mathrm{id}$, so G' is isomorphically mapped by w to a subgroup of D_6. ☐

Lemma 2.4.4. *If S is a G-del Pezzo surface, then the image of G by w in D_6 must contain the subgroup of D_6 isomorphic to $\mathbb{Z}_6$, or the subgroup isomorphic to $\mathfrak{S}_3$ that acts transitively on the (-1)-curves of S.*

Proof. Assume that G does not act transitively on the (-1)-curves of S. Checking all possible subgroups of D_6, we find that one of the divisors $E_1 + E_2 + E_3$, or $\mathrm{D}_{12} + \mathrm{D}_{23} + \mathrm{D}_{13}$, or $E_i + E_{jk}$, with $j,k \neq i$, is invariant by G. In all those cases, according to Castelnuovo's contractibility criterion, there exists a G-birational morphism either to $\mathbb{P}^2$ or to $\mathbb{P}^1 \times \mathbb{P}^1$. ☐

Corollary 2.4.5. *Let G be a subgroup of $\mathrm{Aut}(S)$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Then G is not isomorphic to any of the groups $\mathbb{Z}_2^2, \mathbb{Z}_3, \mathbb{Z}_2, \{\mathrm{id}\}$.*

The only remaining groups of interest are D_6, $\mathbb{Z}_6$, and $\mathfrak{S}_3$. Let us start with the case of $\mathfrak{S}_3$.

Proposition 2.4.6. *Let G be a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_3$ and such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Then, up to conjugation in $\mathrm{Aut}(S)$, the group G is generated by $(x:y:z)\times(u:v:w)\mapsto(y:z:x)\times(v:w:u)$ and $(x:y:z)\times(u:v:w)\mapsto(w:v:u)\times(z:y:x)$. Moreover, the surface S is not G-solid.*

Proof. If the restriction of w to G is not injective, then the image of G by w is isomorphic to $\mathbb{Z}_3$, isomorphic to $\mathbb{Z}_2$, or trivial. By Lemma 2.4.4, it implies that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, contradicting our assumption. Hence $w(G)$ is isomorphic to $\mathfrak{S}_3$ and acts transitively on the (-1)-curves, by Lemma 2.4.4. We deduce that the group G is generated by an element g such that $w(g) = w(r^2)$, and an element h such that $w(h) = w(rs)$. Geometrically, rs acts on the hexagon formed by the (-1)-curves as a reflection that fixes two opposite vertices. By Lemma 2.4.1, we have $g\colon (x:y:z)\times(u:v:w)\mapsto(y:z:x)\times(v:w:u)$ up to conjugation by an element of the torus. Since $\ker(w) = \mathbb{T}$, $h = trs$ for some $t\in\mathbb{T}$. Explicitly, h is of the form $(x:y:z)\times(u:v:w)\mapsto(w:av:bu)\times(z:a^{-1}y:b^{-1}x)$, for some $(a,b)\in(\mathbb{C}^*)^2$. We get $h^2\colon (x:y:z)\times(u:v:w)\mapsto(b^{-1}x:y:bz)\times(bu:v:b^{-1}w)$, and knowing that $\mathrm{Ord}(h) = 2$, we deduce that $b = 1$. Moreover, $hgh\colon (x:y:z)\times(u:v:w)\mapsto(a^{-1}y:az:x)\times(av:a^{-1}w:u)$, but the relations in D_6 imply that $hgh = g^{-1}\colon (x:y:z)\times(u:v:w)\mapsto(y:z:x)\times(v:w:u)$. Hence, $a = 1$, so that $h\colon (x:y:z)\times(u:v:w)\mapsto(w:v:u)\times(z:y:x)$. The three points of the form $(1:\mu^{2k}:\mu^k)\times(1:\mu^k:\mu^{2k})$ are in general position and fixed by the action of G. Blowing up one of them, we get a G-link to the surface $\mathbb{P}^1\times\mathbb{P}^1$ with two fixed points on it in general position. Hence, we are in the case of Example 2.3.1, so that S is G-birational to the G-conic bundle $\mathbb{F}_1$. □

Proposition 2.4.7. *If G is a subgroup of $\mathrm{Aut}(S)$ isomorphic to D_6 such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$, then, up to conjugation in $\mathrm{Aut}(S)$, the group G is generated by $(x:y:z)\times(u:v:w)\mapsto(w:u:v)\times(z:x:y)$ and $(x:y:z)\times(u:v:w)\mapsto(x:z:y)\times(u:w:v)$. Moreover, the surface S is not G-solid.*

Proof. Assume that $G\cong\mathrm{D}_6$. Going through the possible quotients of D_6 and combining with Lemma 2.4.4, we see that either $w(G) = \mathrm{D}_6$, or $w(G) = \mathfrak{S}_3$. We will exclude the latter case, in which $G\cap\ker(w) = T\cong\mathbb{Z}_2$. Since the only extension of the form

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow G \longrightarrow \mathfrak{S}_3 \longrightarrow 1$$

splits, and since $\mathbb{Z}_2$ acts trivially on the Picard group of S, we deduce that the group G has a subgroup H isomorphic to $\mathfrak{S}_3$ such that $\mathrm{Pic}^H(S) = 1$. But such a group is explicitly given in Lemma 2.4.6, and we see that it cannot commute with a subgroup of $\mathbb{T}$ isomorphic to $\mathbb{Z}_2$. Hence we get $G\cong w(G) = \mathrm{D}_6$, and the group G is generated by an element g such that $w(g) = w(r)$, and an element h such that $w(h) = w(s)$. By Lemma 2.4.1, the automorphism g is conjugated to r by an element of $\mathbb{T}$, and its unique fixed point is $P = (1:1:1)\times(1:1:1)$. The isomorphism h is of the form $(x:y:z)\times(u:v:w)\mapsto(x:az:bz)\times(u:a^{-1}w:b^{-1}v)$. Since $\mathrm{Ord}(h) = 2$, we get $b = a^{-1}$. Moreover, $hr^2h\colon (x:y:z)\times(u:v:w)\mapsto(az:x:a^{-2}y)\times(a^{-1}w:$

$u : a^2v)$. But the structure of D_6 implies that $hr^2h = r^{-2}\colon (x : y : z) \times (u : v : w) \mapsto (z : x : y) \times (w : u : v)$, so that $a = 1$. Hence, $h = s\colon (x : y : z) \times (u : v : w) \mapsto (x : z : y) \times (u : w : v)$. This automorphism also fixes P, so that, as described in Remark 2.4.2, there exists a G-link from S to $\mathbb{P}^1 \times \mathbb{P}^1$ centered at P. By Theorem 2.3.21, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is not G'-solid for any subgroup $G' \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ isomorphic to D_6. We conclude that S is not G-solid. □

Proposition 2.4.8. *Let G be a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathbb{Z}_6$ and such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Then, up to conjugation in $\mathrm{Aut}(S)$, the group G is generated by $(x : y : z) \times (u : v : w) \mapsto (w : u : v) \times (z : x : y)$. Moreover,the surface S is not G-solid.*

Proof. First, notice that the restriction of w to G is injective. Indeed, if not, then the image of G by w is isomorphic to $\mathbb{Z}_3$, isomorphic to $\mathbb{Z}_2$, or trivial. By Lemma 2.4.4, it implies that $\mathrm{rk}(\mathrm{Pic}^G(S)) > 1$, contradicting our assumption. The group G is then generated by an element g such that $w(g) = w(r)$. By Lemma 2.4.1, the automorphism g is conjugated to $r\colon (x : y : z) \times (u : v : w) \mapsto (w : u : v) \times (z : x : y)$ by an element of $\mathbb{T}$. The only fixed point of r in S is $(1 : 1 : 1) \times (1 : 1 : 1)$. In particular, there is a subgroup G' of $\mathrm{Aut}(S)$ containing G and isomorphic to D_6. Since S is not G'-solid by Proposition 2.4.7, the surface S is not G-solid either. □

Summing up the results of this section, we get the following.

Theorem 2.4.9. *Let G be a subgroup of $\mathrm{Aut}(S)$ such that $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Then S is G-solid if and only if G is not isomorphic to $\mathbb{Z}_6$, $\mathfrak{S}_3$, or D_6.*

2.5 The projective plane

The only remaining smooth del Pezzo surface is $S = \mathbb{P}^2$, whose automorphism group is $\mathrm{PGL}_3(\mathbb{C})$. The G-rigidity of S has been studied by D.Sakovics in Sakovics (2019). We will point out how his results hold for the G-solidity of S, and describe the full G-birational geometry of S for $G \subset \mathrm{PGL}_3(\mathbb{C})$ isomorphic to $\mathfrak{A}_4$ or $\mathfrak{S}_4$. In other words, we are going to list all G-Mori fibre spaces S' such that there exists a G-birational map $S \dashrightarrow S'$.

Theorem 2.5.1 (Sakovics (2019)). *The projective plane is G-rigid if and only if G is transitive and not isomorphic to $\mathfrak{S}_4$ or $\mathfrak{A}_4$.*

Moreover, if G fixes a point on S, then we can G-equivariantly blow-up this point and get a G-birational map to the G-conic bundle $\mathbb{F}_1$, so that S is not G-solid. Hence, the only remaining cases to study are those of $\mathfrak{S}_4$ and $\mathfrak{A}_4$.

Lemma 2.5.2. *The subgroups of $\mathrm{PGL}_3(\mathbb{C})$ isomorphic to $\mathfrak{S}_4$ or $\mathfrak{A}_4$ are unique up to conjugation in $\mathrm{PGL}_3(\mathbb{C})$.*

Proof. The canonical projection $\pi : \mathrm{GL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$ induces a surjection $\mathrm{SL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$. Let $G \subset \mathrm{PGL}_3(\mathbb{C})$ be a subgroup of $\mathrm{PGL}_3(\mathbb{C})$ isomorphic to $\mathfrak{S}_4$, and consider its lift $G' \subset \mathrm{SL}_3(\mathbb{C})$ by the above projection. The kernel of π restricted to $\mathrm{SL}_3(\mathbb{C})$ is $\{I_3, \mu I_3, \mu^2 I_3\}$, where μ is a primitive cube root of the unity. Thus, we have an extension

$$1 \longrightarrow \mathbb{Z}_3 \longrightarrow G' \longrightarrow G \longrightarrow 1.$$

But $\{I_3, \mu I_3, \mu^2 I_3\}$ lies in the center of $\mathrm{GL}_3(\mathbb{C})$. We deduce that G' is isomorphic to $\mathbb{Z}_3 \times \mathfrak{S}_4$, since this group is the only triple central extension of $\mathfrak{S}_4$. Its subgroup $\{\mathrm{id}\} \times \mathfrak{S}_4$ is sent isomorphically to G by π. In particular, there exists a subgroup of $\mathrm{GL}_3(\mathbb{C})$ isomorphic to $\mathfrak{S}_4$, whose projection in $\mathrm{PGL}_3(\mathbb{C})$ is G. But the only irreducible faithful linear representations of degree 3 of $\mathfrak{S}_4$, up to equivalence of representations, are the standard one and its product with the sign representation. Both are mapped by π to the same subgroup of $\mathrm{PGL}_3(\mathbb{C})$.

The group $\mathfrak{A}_4$ has two triple central extensions, namely $\mathbb{Z}_3 \times \mathfrak{A}_4$, and a non-split extension. But in the second case, there is no irreducible faithful representation of degree 3 whose image is in $\mathrm{SL}_3(\mathbb{C})$. Hence, as in the case of $\mathfrak{S}_4$, there is a subgroup of $\mathrm{GL}_3(\mathbb{C})$ mapped isomorphically by π onto G. Since there is only one equivalence class of irreducible linear representations of $\mathfrak{A}_4$ of degree 3, we conclude that $\mathfrak{A}_4$ is unique in $\mathrm{PGL}_3(\mathbb{C})$, up to conjugation. □

Proposition 2.5.3. *Let $G \cong \mathfrak{S}_4$ be a subgroup of $\mathrm{Aut}(S)$. The only G-links starting from S are:*

- *A link of type I of the form*

$$\begin{array}{ccc} & Z_5 & \\ {\scriptstyle\sigma}\swarrow & \downarrow{\scriptstyle\pi} & \\ S & \mathbb{P}^1 & \end{array}$$

where $\pi : Z_5 \to \mathbb{P}^1$ is a G-conic bundle on a del Pezzo surface of degree 5.

- *A link of type II of the form*

$$\begin{array}{ccc} & Z_6 & \\ {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\tau} \\ S & \overset{\tau}{\dashrightarrow} & S \end{array}$$

where Z_6 is a del Pezzo surface of degree 6, and τ is the standard Cremona involution.

Moreover, the only G-link starting from Z_5 is the inverse of (2.5.3)*, leading back to S.*

Proof. Let G be the subgroup of $\mathrm{PGL}_3(\mathbb{C})$ generated by $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. This group is isomorphic to $\mathfrak{S}_4$. Recall that two isomorphic subgroups of $\mathfrak{S}_4$ are always conjugated in $\mathfrak{S}_4$, so that it is enough to find an occurrence of each subgroup up to isomorphism in the study of the possible stabilizers.

- There is no fixed point under the above action.
- Notice that the subgroup of G generated by A and B is isomorphic to $\mathfrak{A}_4$. It is the only subgroup of G of index two and has no fixed point, so that G does not have any orbit of length 2.
- The only subgroup of index three up to conjugation is D_4, generated by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Its fixed points are $O_3 = \{(1:0:0),(0:1:0),(0:0:1)\}$, and O_3 is the only orbit of length 3 under the action of G. It is the centre of the link of type II:

$$\begin{array}{ccc} & Z_6 & \\ {}^{\sigma}\swarrow & & \searrow^{\tau} \\ S & \overset{\tau}{\dashrightarrow} & S \end{array}$$

  where Z_6 is a del Pezzo surface of degree 6, and τ is the standard Cremona involution.
- The group G does have any orbit of length 4 in general position. Indeed, G has a unique subgroup of index 4. It is isomorphic to $\mathfrak{S}_3$, and generated by $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. The only point fixed by $\mathfrak{S}_3$ is $(1:1:1)$, and its G-orbit is $O_4 = \{(1:1:1),(-1:1:1),(1:-1:1),(1:1:-1)\}$. Hence, there is a G-link of type I of the form:

$$\begin{array}{ccc} & \mathbb{F}_1 & \\ {}^{\sigma}\swarrow & & \downarrow{\scriptstyle\pi} \\ S & & \mathbb{P}^1 \end{array} \tag{1}$$

  where $\pi : X \to \mathbb{P}^1$ is a G-conic bundle on a del Pezzo surface of degree 5, with invariant Picard rank 2.

- There is an orbit of length 6, but not in general position. Indeed, there are two subgroups of index 6 in G. The first one is $\mathbb{Z}_2^2$, generated by $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Its only fixed point is $(0:0:1)$, and its orbit is O_3, which is of length 3. The other subgroup of index 6 of G is Z_4, generated by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Its fixed points are $(0:-i:1)$, $(0:i:1)$, and $(1:0:0)$. The orbit of the last one is O_3, and the two others have the orbit $O_6 = \{(0:i:1),(0:-i:1),(1:0:i),(1:0:-i),(i:1:0),(-i:1:0)$. These six points are not in general position, as they all lie on the Fermat conic $x^2+y^2+z^2=0$.
- There is no orbit of length 8 in general position. The only subgroup of index 8 is $\mathbb{Z}_3$, generated by $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Its fixed points are $(1:1:1)$, $(1:\omega_3:\omega_3^2)$, and $(1:\omega_3^2:\omega_3)$, where ω_3 is a primitive cube root of the unity. The orbit of $(1:1:1)$ is of length 4, and $(1:\omega_3:\omega_3^2)$, and $(1:\omega_3^2:\omega_3)$ lie on the same orbit of length 8: $O_8 = \{(1:\omega_3:\omega_3^2),(1:\omega_3:\omega_3^2),(1:\omega_3:\omega_3^2),(1:\omega_3:\omega_3^2),(1:\omega_3^2:\omega_3),(1:\omega_3^2:\omega_3),(1:\omega_3^2:\omega_3),(1:\omega_3^2:\omega_3)\}$. But these eight points are on the conic $x^2=yz$.

It remains to show that there is no G-link starting from X, except the inverse of the link (1). For this, we will show that all the orbits under the action of G lifted on $\mathbb{F}_1$ have several points on the same fibre of the conic bundle. The birational map γ is given by the linear system $|C|$ of conics passing through all the points of the orbit O_4. The curves $x^2-y^2=0$ and $x^2-z^2=0$ form a basis of this linear system. Hence, up to a change of basis, the map γ is of the form $(x:y:z) \mapsto (x^2-y^2:x^2-z^2)$. The image of G by γ is isomorphic to $\mathfrak{S}_3$. Hence the kernel N of the induced morphism $G \to \mathrm{Aut}(\mathbb{P}^1)$ is isomorphic to $\mathbb{Z}_2^2$, and generated by the matrices $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The action of N on the smooth conics of this system is faithful, hence N does not fix any point on the regular fibres of the conic bundle. □

We get the following immediate consequences.

Corollary 2.5.4. *Let G be a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_4$. The projective plane is not G-solid, but is not G-birational to any Hirzebruch surface.*

Corollary 2.5.5. *Let G be a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_4$. Then $\mathrm{Bir}^G(S) = \langle G, \tau \rangle \cong \mathfrak{S}_4 \times \mathbb{Z}_2$, where τ is the standard Cremona involution.*

Proof. The elements of G are the automorphisms of the form $(x:y:z) \mapsto \sigma(\alpha x:\beta y:\gamma z)$, where σ is a permutation of the coordinates, and $\alpha,\beta,\gamma \in \{-1,1\}$. The involution $\tau:(x:y:z) \dashrightarrow (yz:xz:xy)$ commutes with all these elements. $\square$

The remaining case to study is that of $G \cong \mathfrak{A}_4$.

Proposition 2.5.6. *Let $G \cong \mathfrak{A}_4$ be a subgroup of $\mathrm{Aut}(S)$. The only G-links starting from S are:*

- *Three links of type I of the form*

$$\begin{array}{ccc} & Z_5 & \\ {\scriptstyle\sigma}\swarrow & & \downarrow{\scriptstyle\pi} \\ S & & \mathbb{P}^1 \end{array} \tag{2}$$

  where $\pi: X \to \mathbb{P}^1$ is a G-conic bundle on a del Pezzo surface of degree 5.
- *A link of type II of the form*

$$\begin{array}{ccc} & Z_6 & \\ {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\tau} \\ S & \overset{\tau}{\dashrightarrow} & S \end{array}$$

  where Z_6 is a del Pezzo surface of degree 6, and τ is the standard Cremona involution.
- *A one parameter family of links of type II of the form*

$$\begin{array}{ccc} & Z & \\ {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\tau} \\ S & \overset{i_a}{\dashrightarrow} & S \end{array}$$

  where σ is the blow-up of an orbit of six points, τ is the G-equivariant contraction of eight (-1)-curves, and i_a is a birational involution.

The only G-link starting from X is the inverse of (2)*, leading back to S.*

Proof. Up to conjugation in $\mathrm{PGL}_3(\mathbb{C})$, the group G is generated by the matrices $a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and $b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Recall that any two isomorphic subgroups of $\mathfrak{A}_4$ are conjugated to each other in $\mathfrak{A}_4$.

- There is no fixed point under the above action.
- There is no subgroup of index 2 in G.

- The only subgroup of index 3 is $N = \mathbb{Z}_2^2$, generated by $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Its only fixed point is $(0:0:1)$, whose orbit is $O_3 = \{(1:0:0),(0:1:0),(0:0:1)\}$. The only link centred at this orbit is- the link of type II of the form

$$\begin{array}{ccccc} & & Z & & \\ & \sigma \swarrow & & \searrow \tau & \\ S & - & - \overset{\tau}{-} - & \dashrightarrow & S \end{array}$$

where τ is the standard Cremona involution.

- The only subgroup of G of index 4 is $\mathbb{Z}_3$, generated by $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and have three independant fixed points: $(1:1:1)$, $(1:\omega_3:\omega_3^2)$, and $(1:\omega_3^2:\omega_3)$. They give rise to three distinct orbits of length 4 in general position: $O_4 = \{(1:1:1),(-1:1:1),(1:-1:1),(1:1:-1)\}$, $O_4' = \{(1:\omega_3:\omega_3^2),(-1:\omega_3:\omega_3^2),(1:-\omega_3:\omega_3^2),(1:\omega_3:-\omega_3^2)\}$, and $O_4'' = \{(1:\omega_3^2:\omega_3),(-1:\omega_3^2:\omega_3),(1:-\omega_3^2:\omega_3),(-1:-\omega_3^2:\omega_3)\}$. In each case, the points are in general position, and blowing-up one of them, we get a G-link of type I of the form

$$\begin{array}{ccc} & & Z_5 \\ & \sigma \swarrow & \downarrow \\ S & & \mathbb{P}^1 \end{array} \tag{3}$$

where Z_5 is a del Pezzo surface of degree 5.

- The unique subgroup of index 6 of G is isomorphic to $\mathbb{Z}_2$, generated by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Its fixed points are the points of O_3, and those of the form $(0:1:a)$, with $a \neq 0$. They form orbits of length 6 of the form $O_6^a = \{(0:1:a),(a:0:1),(1:a:0),(0:-1:a),(a:0:-1),(-1:a:0)\}$. The only Sarkisov link centred at such orbit is the link of type II of the form

$$\begin{array}{ccccc} & & Z & & \\ & \sigma \swarrow & & \searrow \tau & \\ S & - & - \overset{i_a}{-} - & \dashrightarrow & S \end{array}$$

where σ is the blow-up of O_6^a, τ is the G-equivariant contraction of eight (-1)-curves, and i_a is the birational involution $(x:y:z) \mapsto (f_1(x,y,z) : f_2(x,y,z) : f_3(x,y,z))$, where

$$\begin{aligned} f_1(x,y,z) &= \left(a^{12}+1\right)x^2y^2z + a^{10}\left(-y^4\right)z + 2a^8y^2z^3 - a^6z^5 + 2a^4x^2z^3 - a^2x^4z, \\ f_2(x,y,z) &= \left(a^{12}+1\right)x^2yz^2 - a^{10}x^4y + 2a^8x^2y^3 - a^6y^5 + 2a^4y^3z^2 - a^2yz^4, \text{ and} \\ f_3(x,y,z) &= \left(a^{12}+1\right)xy^2z^2 + a^{10}(-x)z^4 + 2a^8x^3z^2 - a^6x^5 + 2a^4x^3y^2 - a^2xy^4. \end{aligned}$$

We will now show that there is no G-link starting from any of the G-conic bundles X_1, X_2, and X_3 of degree 5, except the inverse of the link (3). The G-conic bundle X_1 is the same as X in the proof of Proposition 2.5.3, and the proof is the same. The G-conic bundle X_2 is the blow-up of S in the points of $O_4' = \{(1:\omega_3:\omega_3^2), (-1:\omega_3:\omega_3^2), (1:-\omega_3:\omega_3^2), (1:\omega_3:-\omega_3^2)\}$. The linear system of conics passing through these points is generated by $\mu x^2 - z^2 = 0$ and $(\mu+1)x^2 + y^2 = 0$. The G-conic bundle X_3 is the blow-up of S in the points of $O_4' = \{(1:\omega_3^2:\omega_3), (-1:\omega_3^2:\omega_3), (1:-\omega_3^2:\omega_3), (1:\omega_3^2:-\omega_3)\}$. The linear system of conics passing through these points is generated by $\mu x^2 - y^2 = 0$ and $(\mu+1)x^2 + z^2 = 0$. In both of these cases, and the subgroup $N \cong \mathbb{Z}_2^2$ of G acts faithfully on each smooth conic of the system, hence does not fix any point in the fibres of the conic bundle. □

Once again, we get the following consequences.

Corollary 2.5.7. *Let G be a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{A}_4$. The projective plane is not G-solid, but not G-birational to any Hirzebruch surface.*

Corollary 2.5.8. *Let G be a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{A}_4$. Then $\mathrm{Bir}^G(S) = \langle G, \tau, i_a | a \in \mathbb{C}^* \rangle$, where τ is the standard Cremona involution.*

Summing up the results of this section, we can conclude about the G-solidity of the projective plane.

Theorem 2.5.9. *Let G be a finite subgroup of $\mathrm{PGL}_3(\mathbb{C})$. The following assertions are equivalent.*

- *The projective plane is G-rigid,*
- *The projective plane is G-solid,*
- *The group G is transitive and not isomorphic to $\mathfrak{S}_4$ or $\mathfrak{A}_4$.*

Chapter 3

Linearization problem for finite subgroups of the plane Cremona group

"A speaker stops getting nervous only when they start giving bad talks."

Lucy Moser-Jauslin

We give a complete solution to the linearization problem in the plane Cremona group over the field of complex numbers. The results presented in this chapter have been obtained in collaboration with Egor Yasinsky and Arman Sarikyan, see Pinardin et al. (2024). All authors have approved the inclusion of this work in the present thesis and acknowledge equal contribution.

3.1 Projective linearizability and linearizability, an update.

This section updates Pinardin et al. (2024). In accordance with the Academic Quality and Standards of the University of Edinburgh[1], we use this introductory part for it and leave the body of the paper unchanged. To avoid overloading terminology and to remain aligned with our main focus, namely the conjugacy in the Cremona group, we chose to define as linearizable a subgroup of $\mathrm{Cr}_n(\mathbb{C})$ that is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^n)$, or, equivalently, a group acting faithfully on a variety which is equivariantly rational. In fact, many authors define the linearizability as something more restrictive. We present this definition here and apply it only until Corollary 3.1.3.

1. https://registryservices.ed.ac.uk/academic-services/students/thesis-submission

Definition 3.1.1. A subgroup G of $\mathrm{Cr}_n(\mathbb{C})$ that is conjugate to a subgroup H of $\mathrm{Aut}(\mathbb{P}^2)$ is called *projectively linearizable*. If, in addition, the group H admits an isomorphic lift to a subgroup of $\mathrm{GL}_{n+1}(\mathbb{C})$, then G is called *linearizable*.

While the former notion is the most natural for our purposes, it may nevertheless be reasonable to add the condition of isomorphic lift to the linear group, as above. For example, a linear representation can always be naturally extended to a linear representation of higher dimension. Thus one might expect the same kind of extendability for linearizable actions. Yet a projectively linearizable group G faithfully acting on a rational variety X need not induce a projectively linearizable action of G on the product $X \times \mathbb{P}^n$. This becomes true if G is linearizable in the sense of definition 3.1.1.

The main point of this addendum to Pinardin et al. (2024) is that we recently found out that, on surfaces, projective linearizability and linearizability coincide for all actions except those that are trivially linearizable, that is, when G is a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ without an isomorphic lift to $\mathrm{GL}_3(\mathbb{C})$. The following two statements are corollaries of Theorem 3.2.1 and its proof.

Corollary 3.1.2. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a finite subgroup. Consider a regularization of G on a smooth rational G-minimal surface S. Then G is* linearizable *if and only if there exists a G-birational map $\varphi\colon S \dashrightarrow \mathbb{P}^2$ such that $\varphi G \varphi^{-1}$ fixes a point on $\mathbb{P}^2$.*

Proof. It follows from the proof of Theorem 3.2.1, via a case-by-case study of the subgroups of $\mathrm{Aut}(\mathbb{P}^2)$ that are obtained by birational conjugation of projectively linearizable subgroups of $\mathrm{Cr}_2(\mathbb{C})$. □

Corollary 3.1.3. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a finite subgroup. Consider a regularization of G on a smooth rational G-minimal surface S. Assume moreover that $S \neq \mathbb{P}^2$. Then G is linearizable if and only if it is projectively linearizable.*

Proof. Assume that G is linearizable. Then, according to 3.1.2, there exists a G-birational map $\varphi\colon S \dashrightarrow \mathbb{P}^2$ such that $G' = \varphi G \varphi^{-1}$ fixes a point P on $\mathbb{P}^2$. Hence, the group G acts faithfully on $\mathbb{C}^2$, the tangent space of $\mathbb{P}^2$ at P. □

We now conclude this remark and revert to Definition 1.5.1 for the notion of linearizability. Let us present the results of Pinardin et al. (2024).

3.2 Classification of linearizable actions

In this paper, we use the extremely powerful *Sarkisov program*, which was introduced in Subsection 1.6.1. It has enabled us to prove that the "majority" of subgroups of the plane Cremona group are non-linearizable, while the linearizable ones belong to the following compact list.

Main Theorem 3.2.1. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a finite subgroup. Consider a regularization of G on a G-minimal two-dimensional G-Mori fibre space S over the base B. Then G is* linearizable *if and only if the pair (S,G) is one of the following*[2]*:*

K_S^2	*Surface S*	*Group $G \subset \mathrm{Aut}(S)$*	*Reference*
		G-conic bundles $(over\ B \simeq \mathbb{P}^1)$	
$K_S^2 = 8$	*A Hirzebruch surface $\mathbb{F}_n$ with n odd*	— *any*	*Theorem 3.6.1*
$K_S^2 = 8$	*A Hirzebruch surface $\mathbb{F}_n$ with $n > 0$ even*	— *acts cyclically on B* — *acts as D_{2m+1} on B*	*Theorem 3.6.1*
$K_S^2 = 8$	*The quadric surface $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$*	— $\mathbb{Z}_n \times_Q \mathbb{Z}_m$ — $\mathbb{Z}_n \times_Q \mathrm{D}_{2m+1}$ — $\mathrm{D}_{2n+1} \times_Q \mathrm{D}_{2m+1}$ *is dihedral*	*Theorem 3.6.12*
		G-del Pezzo surfaces $(over\ B = \mathrm{pt})$	
$K_S^2 = 5$	*The unique quintic del Pezzo surface*	— $\mathbb{Z}_5$ — D_5	*Proposition 3.4.3*
$K_S^2 = 6$	*The unique sextic del Pezzo surface*	— $\mathbb{Z}_6$ — $\mathfrak{S}_3$	*Proposition 3.4.14*
$K_S^2 = 8$	*The quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$*	— $(\mathbb{Z}_n \times_Q \mathbb{Z}_n)_{\bullet}\mathbb{Z}_2$	*Proposition 3.6.7*
$K_S^2 = 9$	*The projective plane $\mathbb{P}^2$*	— *Blichfeldt's list*	*Section 1.7.2*

This paper is organized as follows. It employs a significant amount of (elementary) finite group theory, so relevant facts are isolated in Section 1.7. In Section 3.4, we derive part of the main theorem related to G-del Pezzo surfaces. Although Section 3.5 is not used later on, it aims to fill a gap in the literature by explicitly describing, in matrix terms, finite groups acting on smooth two-dimensional quadrics (equivalently, finite subgroups of the projective orthogonal group $\mathrm{PO}(4)$). In Section 3.6, we examine the linearizability of finite groups acting on Hirzebruch surfaces, which finishes the proof of our main result. Finally, in the Appendix we provide the supporting Magma code for Section 3.5 (note that this code is for the reader's convenience only and essentially is not used in any proof).

2. We refer to Notations 1.7.1 for the group-theoretic notations.

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3.3 G-minimal surfaces

So far we have reduced the problem of classification of finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$ to classification of G-Mori fibre spaces up to G-birational equivalence. Note that G-conic bundles $\pi\colon S\to B$ do not have to be G-minimal in the absolute sense, i.e. it is not necessarily true that every G-birational morphism $S\to T$ is G-isomorphism (a trivial example is the blow-up $\mathbb{F}_1\to\mathbb{P}^2$ of a G-fixed point on $\mathbb{P}^2$). However, we have a precise description of all such cases. All results below are essentially due to V. Iskovskikh.

Theorem 3.3.1 ((Iskovskih, 1979, Theorems 4 and 5))**.** *Let $\pi\colon S\to\mathbb{P}^1$ be a G-conic bundle.*

1. *If S is not G-minimal, then S is a del Pezzo surface (in particular, $K_S^2\geqslant 1$).*
2. *Assume that $1\leqslant K_S^2\leqslant 8$. Then S is G-minimal if and only if $K_S^2\in\{1,2,4,8\}$.*
3. *If $K_S^2\in\{1,2,4\}$ then S is a del Pezzo surface if and only if there are exactly two G-conic bundle structures on S.*

Lemma 3.3.2 (see (Iskovskih, 1979, Theorem 5))**.** *Let S be a rational G-surface with $K_S^2=7$. Then S is not a G-Mori fibre space. In other words, S is neither a G-del Pezzo surface, nor a G-conic bundle.*

For a G-conic bundle $\pi\colon S\to\mathbb{P}^1$, Noether's formula implies that $K_S^2=8-c$, where c is the number of singular fibres of π. In particular, $K_S^2\leqslant 8$. Besides, Theorem 1.6.2 says that the G-conic bundles with $c\geqslant 8$ are G-birationally superrigid and hence are not G-birational to $\mathbb{P}^2$. For our purposes, we may assume that S is also G-minimal (in the absolute sense), hence $K_S^2\in\{1,2,4,8\}$ by Theorem 3.3.1.

Proposition 3.3.3. *Let $\pi\colon S\to\mathbb{P}^1$ be a G-conic bundle. If $K_S^2\in\{1,2,4\}$, then G is not linearizable.*

Proof. Indeed, S is G-minimal by Theorem 3.3.1. In particular, it does not admit Sarkisov links of type III, i.e. contractions to G-del Pezzo surfaces. Hence, all Sarkisov G-links starting from S are of type II (elementary transformations) or IV and do not change K_S^2. Therefore, G is not linearizable. □

Summary. So, it remains to investigate the linearizability in the following two cases:

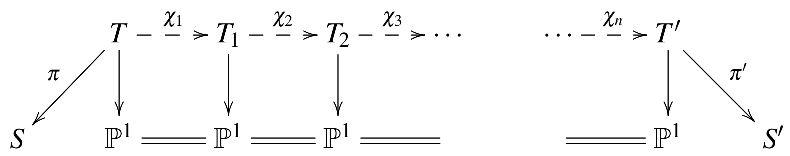
— S is a G-del Pezzo surface, where we can assume that $K_S^2\geqslant 4$ by Theorem 1.6.1. If $K_S^2=8$, we can assume that S is not the blow-up of $\mathbb{P}^2$ at a G-fixed point, because such S is not G-minimal; for the same reason, we skip the case $K_S^2=7$, see Lemma 3.3.2.

— S is a G-conic bundle $\pi\colon S\to\mathbb{P}^1$ with no singular fibres, i.e. S is a Hirzebruch surface $\mathbb{F}_n$ acted on by a finite group G.

3.4 G-del Pezzo surfaces

3.4.1 Degree 4

By Theorem 1.6.1, none of the groups G that act biregularly on a del Pezzo surface S with $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$ and $K_S^2 \leqslant 3$ is linearizable. Let S be a G-del Pezzo surface of degree 4 and $\varphi\colon S \dashrightarrow S'$ be a G-birational map to another G-del Pezzo surface S'. Then it follows from the classification of Sarkisov G-links that φ, if not an isomorphism, fits into the commutative diagram of G-maps



(1)

where π is the blow-up of a G-fixed point on S, which leads to a cubic surface T equipped with a structure of a G-conic bundle (so π is a link of type I); the maps χ_i are elementary transformations of G-conic bundles (links of type II), and π' is the blow-down of a G-orbit of (-1)-curves, where S' is again a G-del Pezzo surface of degree 4. We conclude that none of such groups G is linearizable.

3.4.2 Degree 5

Let S be a G-del Pezzo surface of degree 5. Recall that there is a single isomorphism class of del Pezzo surfaces of degree 5 over $\mathbb{C} = \overline{\mathbb{C}}$. Every Sarkisov G-link starting from S is of type II, where η blows up a G-orbit of length d, and one of the following holds:

1. $S \simeq S'$, $d = 4$, χ is a birational Bertini involution;
2. $S \simeq S'$, $d = 3$, χ is a birational Geiser involution;
3. $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $d = 2$;
4. $S' \simeq \mathbb{P}^2$, $d = 1$.

The following proposition lists all possible groups G.

Proposition 3.4.1 ((Dolgachev & Iskovskikh, 2009, Theorem 6.4))**.** *Let S be a del Pezzo surface of degree 5 and $G \subset \mathrm{Aut}(S)$ be a group such that $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. Then G is isomorphic to one of the following five groups:*

$$\mathfrak{S}_5,\ \mathfrak{A}_5,\ \mathrm{F}_5,\ \mathrm{D}_5,\ \mathbb{Z}_5,$$

where $\mathrm{F}_5 = \langle a, b \mid a^5 = b^4 = 1, bab^{-1} = a^3\rangle$ is the Frobenius group[3] of order 20.

Proposition 3.4.2 ((Cheltsov, 2008, Example 6.3), (Cheltsov, 2014, Theorem B.10))**.** *The del Pezzo surface of degree 5 is $\mathfrak{A}_5$- and $\mathfrak{S}_5$-birationally superrigid.*

3. GAP ID [20,3]

Proof. None of these two groups has faithful 2-dimensional representations over $\mathbb{C}$, hence none of them has an orbit of size 1 or 2 on our surface: recall, see (Bialynicki-Birula, 1973, Lemma 2.4), that if X is an irreducible algebraic variety and $G \subset \mathrm{Aut}(X)$ is a finite group fixing a point $p \in X$ then the induced linear representation $G \hookrightarrow \mathrm{GL}(T_pX)$ is faithful. Furthermore, these groups have no subgroups of index 3 or 4, hence there are no equivariant birational Bertini or Geiser involutions on (with respect to these groups). The result follows. □

Proposition 3.4.3. *Let S be a G-del Pezzo surface of degree 5. Then G is linearizable if and only if $G \simeq \mathbb{Z}_5$ or $G \simeq \mathrm{D}_5$.*

Proof. Proposition 3.4.2 implies that the groups $\mathfrak{A}_5$ and $\mathfrak{S}_5$ are not linearizable; in fact, $\mathfrak{S}_5$ is not a subgroup of $\mathrm{PGL}_3(\mathbb{C})$. The Frobenius group F_5 is not a subgroup of $\mathrm{PGL}_3(\mathbb{C})$ either: according to the description given in Section 1.7.2, it is obviously not transitive, and moreover has no faithful 2-dimensional representations (in fact, by (Wolter, 2018, Theorem 1.1), there exists a unique F_5-del Pezzo surface which is F_5-birational to S, namely $\mathbb{P}^1 \times \mathbb{P}^1$).

We now show that the groups $\mathbb{Z}_5$ and D_5 are linearizable. Let G be any of these groups. It is enough to construct an explicit G-equivariant map $\mathbb{P}^2 \dashrightarrow S$, where S is a G-del Pezzo surface of degree 5, because there is a single G-isomorphism class of such surfaces. Indeed, there is a single conjugacy class in $\mathrm{Aut}(S) \simeq \mathfrak{S}_5$ of $\mathbb{Z}_5$ and D_5, so any isomorphism $S \xrightarrow{\sim} S'$ to the del Pezzo surface S' of degree 5 can be made a G-isomorphism, after composing with a suitable automorphism of S'. Now, to construct an explicit G-birational map $\mathbb{P}^2 \dashrightarrow S$, one can take two linear automorphisms

$$r\colon [x:y:z] \mapsto [x:\omega_5 y:\omega_5^{-1} z], \quad s\colon [x:y:z] \mapsto [x:z:y],$$

and let $G_1 = \langle r \rangle \simeq \mathbb{Z}_5$, $G_2 = \langle r,s \rangle \simeq \mathrm{D}_5$. The G_1- and G_2-orbit of the point $[1:1:1]$ is a set of 5 points in general position on $\mathbb{P}^2$, lying on the unique smooth conic $Q \subset \mathbb{P}^2$. By blowing up these points and contracting the strict transform of Q we get a G_1-birational (respectively, G_2-birational) map from $\mathbb{P}^2$ to S. □

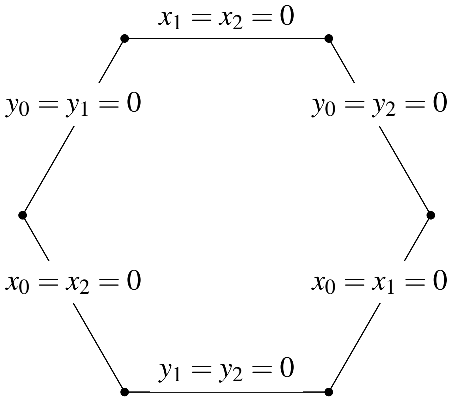
Remark 3.4.4. The actions of $\mathfrak{A}_5$ and $\mathfrak{S}_5$ are stably linearizable. More precisely, by (Y. G. Prokhorov, 2010, Proposition 4.7), Prokhorov shows that $S \times \mathbb{P}^1$ is $\mathfrak{S}_5$-birational to the Segre cubic threefold $\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0$, which is obviously acted on by $\mathfrak{S}_6$. Recall that, up to conjugation, the group $\mathfrak{S}_6$ has two subgroups isomorphic to $\mathfrak{S}_5$, given by the standard and the non-standard embeddings (which differ by an outer automorphism of $\mathfrak{S}_6$). In the construction above, $S \times \mathbb{P}^1$ turns out to be $\mathfrak{S}_5$-birational to the Segre cubic with a non-standard action of $\mathfrak{S}_5$, which is linearizable. Recently, B. Hassett and Yu. Tschinkel gave another proof of this fact using the equivariant torsor formalism, see (Hassett & Tschinkel, 2023, §8.2).

3.4.3 Degree 6

Let S be a del Pezzo surface of degree $K_S^2 = 6$. Then S is the blow-up $\pi\colon S \to \mathbb{P}^2$ in three non-collinear points p_1, p_2, p_3, which we may assume to be $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$, respectively. The surface S can be given as

$$\left\{([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 :\ x_0y_0 = x_1y_1 = x_2y_2\right\}. \tag{2}$$

The set of (-1)-curves on S consists of six curves: the exceptional divisors of blow-up $e_i = \pi^{-1}(p_i)$ and the strict transforms of the lines d_{ij} passing through $p_i,\ p_j$. In the anticanonical embedding $S \hookrightarrow \mathbb{P}^6$ these (-1)-curves form a hexagon Σ; the configuration of lines is shown in the diagram (3) below.



(3)

Note that Σ is naturally acted on by $\mathrm{Aut}(S)$, so there is a homomorphism

$$\Phi\colon \mathrm{Aut}(S) \to \mathrm{Aut}(\Sigma) \simeq \mathrm{D}_6 = \langle r, s \mid r^6 = s^2 = \mathrm{id},\ srs^{-1} = r^{-1}\rangle,$$

where r is a rotation by $\pi/3$ and s is a reflection. The kernel of Φ is the maximal torus $\mathbb{T} \subset \mathrm{PGL}_3(\mathbb{C})$, isomorphic to $(\mathbb{C}^*)^3/\mathbb{C}^* \simeq (\mathbb{C}^*)^2$, and acts on S by

$$(\lambda_0, \lambda_1, \lambda_2) \cdot ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) = ([\lambda_0x_0 : \lambda_1x_1 : \lambda_2x_2], [\lambda_0^{-1}y_0 : \lambda_1^{-1}y_1 : \lambda_2^{-1}y_2]). \tag{4}$$

The action of $\mathbb{T}$ on $S \setminus \Sigma$ is faithful and transitive, and the automorphism group of $\mathrm{Aut}(S)$ fits into the short exact sequence

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Aut}(S) \xrightarrow{\ \Phi\ } \mathrm{D}_6 \longrightarrow 1$$

with $\Phi(\mathrm{Aut}(S)) \simeq \mathrm{D}_6 \simeq \mathfrak{S}_3 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the lift of the standard quadratic involution, acting by

$$\iota\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([y_0 : y_1 : y_2], [x_0 : x_1 : x_2]), \tag{5}$$

and $\mathfrak{S}_3$ acts naturally by permutations on each of the two triples x_0, x_1, x_2 and y_0, y_1, y_2. In what follows, we will denote

$$\theta\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_1 : x_2 : x_0], [y_1 : y_2 : y_0]) \tag{6}$$

the generator of $\mathbb{Z}_3 \subset \mathfrak{S}_3$. Then D_6 is generated by the element $\zeta = \theta \circ \iota$ of order 6 (rotation) and the element σ of order 2 (reflection), whose actions are given by

$$\zeta\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([y_1 : y_2 : y_0], [x_1 : x_2 : x_0]), \tag{7}$$

$$\sigma\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([y_0 : y_2 : y_1], [x_0 : x_2 : x_1]). \tag{8}$$

Every Sarkisov link $\chi\colon S \dashrightarrow S'$ is of type II and starts with blowing up a G-orbit of length d. Then one of the following holds:

1. $S \simeq S'$, $d = 5$, χ is a birational Bertini involution;
2. $S \simeq S'$, $d = 4$, χ is a birational Geiser involution;
3. $d = 3$, $K_{S'}^2 = 6$;
4. $d = 2$, $K_{S'}^2 = 6$;
5. $d = 1$, $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Remark 3.4.5. In cases (3) and (4), the surfaces S' *does not* have to be G-isomorphic to S, see (Yasinsky, 2023, Example 3.9).

Lemma 3.4.6 ((Yasinsky, 2023, Lemma 3.7))**.** *Let S be a G-del Pezzo surface of degree 6. Then G is of the form*

$$T_{\bullet}\langle r\rangle \simeq T_{\bullet}\mathbb{Z}_6, \qquad T_{\bullet}\langle r^2, s\rangle \simeq T_{\bullet}\mathfrak{S}_3, \quad \textit{or} \quad T_{\bullet}\langle r, s\rangle \simeq T_{\bullet}\mathrm{D}_6,$$

where $T \simeq \mathbb{Z}_n \times \mathbb{Z}_m$ is a subgroup of $\mathbb{T}$.

Remark 3.4.7. The dihedral group $\mathrm{D}_6 = \langle r, s\rangle$ contains two groups isomorphic to $\mathfrak{S}_3$, but only for one of them the G-invariant Picard number is 1, namely for $\langle r^2, s\rangle$, which we denote $\mathfrak{S}_3^{\mathrm{m}in}$. The group $\langle r^2, rs\rangle$ will be denoted by $\mathfrak{S}_3^{\mathrm{n}min}$. This is the quotient of D_6 by its centre $\mathrm{Z}(\mathrm{D}_6) \simeq \mathbb{Z}_2$.

The following lemma shows that we do not lose generality by choosing particular actions of the groups $\mathbb{Z}_6$, $\mathfrak{S}_3$ and D_6.

Lemma 3.4.8. *Let S be a G-del Pezzo surface of degree 6, and let G be isomorphic to $\mathbb{Z}_6$, $\mathfrak{S}_3$ or D_6. Then G is conjugate in $\mathrm{Aut}(S)$ to the groups $\langle\zeta\rangle$, $\langle\zeta^2, \sigma\rangle$ and $\langle\zeta, \sigma\rangle$, respectively.*

Proof. This is the content of (Pinardin, 2024, Propositions 5.6, 5.7, 5.8). We give a short direct proof here for the largest of these groups $G \simeq \mathrm{D}_6$. We may assume that it is generated by two elements $\overline{\zeta}$ and $\overline{\sigma}$ whose images in $\mathrm{Aut}(\Sigma)$ are as in (7). The map $\overline{\zeta}$ is given by

$$\overline{\zeta}\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([uy_1 : vy_2 : y_0], [u^{-1}x_1 : v^{-1}x_2 : x_0]) \tag{9}$$

for some $u, v \in \mathbb{C}^*$. The map

$$\beta\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_0 : v^{-1}x_1 : uv^{-1}x_2], [y_0 : vy_1, vu^{-1}y_2])$$

then conjugates $\overline{\zeta}$ to ζ. After this conjugation, $\overline{\sigma}$ is given by

$$\overline{\sigma}\colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([ay_0 : by_2 : y_1], [a^{-1}x_0 : b^{-1}x_2 : x_1])$$

for some $a, b \in \mathbb{C}^*$. The condition $\overline{\sigma}^2 = \mathrm{id}$ implies $b = 1$, while the condition $\overline{\sigma} \circ \zeta \circ \overline{\sigma} = \zeta^{-1}$ implies $a = 1$. □

Lemma 3.4.9. *Let S be a G-del Pezzo surface of degree 6. Then G fixes a point on S if and only if $G \cap \mathbb{T} = \{\mathrm{id}\}$.*

Proof. Assume that G fixes a point on S, but $G \cap \mathbb{T} \neq \{\mathrm{id}\}$. Since $\mathbb{T}$ can be identified with a subgroup of $\mathrm{PGL}_3(\mathbb{C})$ which fixes 3 points on $\mathbb{P}^2$, an element of $\mathbb{T}$ fixing a point on $S \setminus \Sigma$ is necessarily trivial. Therefore, a fixed point $p \in S$ of G lies on Σ. But this implies $\mathrm{rk}\,\mathrm{Pic}(S)^G > 1$.

Conversely, suppose that $G \cap \mathbb{T} = \{\mathrm{id}\}$. Then G is isomorphic to $\mathbb{Z}_6$, $\mathfrak{S}_3$ or D_6 by Lemma 3.4.6. It remains to apply Lemma 3.4.8 and notice that the groups mentioned there fix the point $([1 : 1 : 1], [1 : 1 : 1]) \in S$. □

Lemma 3.4.10. *Let $\varphi\colon S_1 \dashrightarrow S_2$ be a G-birational map of G-del Pezzo surfaces of degree 6. Then G fixes a point on S_1 if and only if G fixes a point on S_2.*

Proof. It is enough to prove the necessity. Let $\varphi\colon (S_1, G, \iota_1) \dashrightarrow (S_2, G, \iota_2)$ be a G-birational map, and assume that $\iota_1(G)$ fixes a point on S_1. By Lemma 3.4.9, the group $\iota_1(G)$ does not intersect the torus $\mathrm{Aut}^\circ(S_1)$ and hence is isomorphic to $\mathbb{Z}_6$, $\mathfrak{S}_3$ or D_6 by Lemma 3.4.6. But this implies that $\iota_2(G)$ does not intersect the torus $\mathrm{Aut}^\circ(S_2)$: otherwise $\iota_2(G)/(\iota_2(G) \cap \mathrm{Aut}^\circ(S_2))$ is isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2^2, \mathfrak{S}_3^{\mathrm{n}min}$ (see Remark 3.4.7), hence S_2 is not a G-del Pezzo surface. □

We now discuss (non)linearizability of three particular groups: $\langle \zeta \rangle \simeq \mathbb{Z}_6$, $\langle \zeta^2, \sigma \rangle \simeq \mathfrak{S}_3$ and $\langle \zeta, \sigma \rangle \simeq \mathrm{D}_6$.

Example 3.4.11. The cyclic group $G = \langle \zeta \rangle \simeq \mathbb{Z}_6$ is linearizable. Indeed, the Sarkisov G-link centred at the G-fixed point $([1 : 1 : 1], [1 : 1 : 1]) \in S$ leads to the G-del Pezzo surface $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Since G is cyclic, it fixes a point on S' (of course, this can be seen directly as in the next example), hence the stereographic projection from it linearizes the action of G.

Example 3.4.12. The symmetric group $G = \langle \zeta^2, \sigma \rangle \simeq \mathfrak{S}_3$ is linearizable as well. Indeed, after the same G-link at the fixed point $([1:1:1],[1:1:1])$, we end up with $S' = \mathbb{P}^1 \times \mathbb{P}^1$ acted on by G. Recall that $\mathrm{Aut}(S') \simeq (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) \rtimes \mathbb{Z}_2$. Since $\mathrm{Pic}(S')^G = 1$, the group G maps non-trivially to the $\mathbb{Z}_2$ factor of this semi-direct product, i.e. $G = \langle r, s \rangle$, where r is an automorphism of order 3 acting fibrewisely on S' and s is an involution which does not preserve the rulings of S'. It will be shown in Proposition 3.6.7 that such a group fixes a point on S', and the stereographic projection from this point linearizes the action of G.

Example 3.4.13 (*Iskovskikh's example*)**.** Answering the question of V. Popov, V. Iskovskikh showed in Iskovskikh (2008) that the group $G \simeq \mathrm{D}_6$ generated by the automorphisms ζ and σ is not linearizable. The proof relied on the classical Sarkisov theory. Recently, another proof was given by in (Hassett et al., [2021] ©2021, Section 7.6) by using the Burnside group $\mathrm{Burn}_2(G)$ and later using the combinatorial Burnside groups, see Tschinkel, Yang, and Zhang (2022). We refer to Section 1.6.3 for more information.

We are ready to summarize the results of this section.

Proposition 3.4.14. *Let S be a G-del Pezzo surface of degree 6. Then G is linearizable if and only if $G \simeq \mathbb{Z}_6$ or $G \simeq \mathfrak{S}_3$.*

Proof. The sufficiency follows from Example 3.4.11, Example 3.4.12 and Lemma 3.4.8. To prove the necessity, we use the classification of G-links given above. Assume that G is linearizable. Since Bertini and Geiser involutions lead to a G-isomorphic surface, there exists a sequence of G-links $S \dashrightarrow S_1 \dashrightarrow \ldots \dashrightarrow S_n = S'$, where all S_i are G-del Pezzo surfaces of degree 6 and G fixes a point on S' — so that we construct a G-link to $\mathbb{P}^1 \times \mathbb{P}^1$. But then G fixes a point on S by Lemma 3.4.10. By Lemma 3.4.9, G does not intersect the torus $\mathrm{Aut}^\circ(S)$, and hence G is mapped isomorphically to a subgroup of D_6. Now the result follows from Lemma 3.4.8 and Examples 3.4.11, 3.4.12 and 3.4.13. □

Remark 3.4.15 (*Cayley groups and stable linearizability*)**.** The (non)linearizability of D_6 and its subgroups was addressed in a different context by N. Lemire, V. Popov and Z. Reichstein in their seminal article, see Lemire, Popov, and Reichstein (2006). They called a connected linear algebraic group G over a field $\mathbf{K}$ a *Cayley group* if it admits a *Cayley map*, i.e. a G-equivariant birational isomorphism between the group variety G and its Lie algebra $\mathrm{Lie}(G)$. Let

$$T = \left\{ A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \ : \ \det A = 1 \right\}, \quad \mathfrak{t} = \left\{ A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \ : \ \mathrm{tr} A = 0 \right\}$$

be a maximal torus of $\mathrm{SL}_3(\mathbb{C})$ and its Lie algebra. Both are obviously acted on by the group $W = \mathfrak{S}_3$ permuting a_1, a_2 and a_3. Furthermore, T admits a natural compactification $\{x_0y_0z_0 = x_1y_1z_1\} \subset \operatorname{Proj}\mathbb{C}[x_0,x_1] \times \operatorname{Proj}\mathbb{C}[y_0,y_1] \times \operatorname{Proj}\mathbb{C}[z_0,z_1]$, which is a W-del Pezzo surface of degree 6. By Example 3.4.12, and as was also shown in (Lemire et al., 2006, 9.1), T and $\mathfrak{t}$ are W-birational. By the Corollary of (Lemire et al., 2006, Lemma 3.5), this is equivalent to showing that $\mathrm{SL}_3(\mathbb{C})$ is a Cayley group, because W is the Weyl group of $\mathrm{SL}_3(\mathbb{C})$.

By the same principle, $\mathbb{G}_2$ is *not* Cayley: its Weyl group is $W = \mathrm{D}_6$, the maximal torus and its Lie algebra are W-isomorphic to T and $\mathfrak{t}$, respectively. But the latter two are not W-birational, as showed in Iskovskikh's Example 3.4.13. Interestingly enough, by (Lemire et al., 2006, Proposition 9.1) $\mathbb{G}_2 \times \mathbb{G}_m^2$ *is* a Cayley group: the varieties $T \times \mathbb{A}^2$ and $\mathfrak{t} \times \mathbb{A}^2$ are D_6-birational; in other words, D_6 is *stably linearizable*. It has recently been shown that one could replace $\mathbb{A}^2$ with $\mathbb{A}^1$ here, see (Böhning, Graf von Bothmer, & Tschinkel, 2023, Proposition 12).

Remark 3.4.16. According to Proposition 3.4.14, any finite group G with $G \cap \mathbb{T} \neq \{\mathrm{id}\}$ is not linearizable. In Hassett and Tschinkel (2023), the authors give an example of stably linearizable $\mathfrak{S}_4$-action on the sextic del Pezzo surface (here, one has $G \cap \mathbb{T} \simeq \mathrm{V}_4$).

3.5 Finite groups acting on smooth quadric surfaces

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. Denote by σ the involution

$$\begin{aligned} S &\to S \\ (x,y) &\mapsto (y,x). \end{aligned}$$

The connected component $\mathrm{Aut}(S)^\circ$ of the identity is the subgroup of the automorphisms, which preserve both rulings. This subgroup is isomorphic to $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$. Then there is a short exact sequence

$$1 \longrightarrow \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}) \longrightarrow \mathrm{Aut}(S) \xrightarrow{\ \psi\ } \langle\sigma\rangle \longrightarrow 1, \tag{10}$$

which splits as the semidirect product $\mathrm{Aut}(S) \simeq (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})) \rtimes \langle\sigma\rangle$, with $\langle\sigma\rangle \simeq \mathbb{Z}_2$ acting on $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ by permuting two factors.

Proposition 3.5.1. *Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $G \subset \mathrm{Aut}(S)$.*

1. *If $\operatorname{rk}\operatorname{Pic}(S)^G \simeq \mathbb{Z}$, then G fits into the short exact sequence*

$$1 \longrightarrow H \times_Q H \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1, \tag{11}$$

   where H is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$.

2. *If $\operatorname{rk}\operatorname{Pic}(S)^G \simeq \mathbb{Z}^2$, then $G \simeq H_1 \times_Q H_2$, where H_1, H_2 are finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$.*

Proof. Let us restrict the exact sequence (10) to G. Clearly, $\psi(G) = \{\mathrm{id}\}$ if and only if G preserves both rulings of S, which is equivalent to $\mathrm{Pic}(S)^G \simeq \mathbb{Z}^2$. If this is the case, then G is a subgroup of $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ and the claim follows from Goursat's lemma 1.7.8 and Proposition 1.7.2.

Suppose that $\psi(G) = \langle\sigma\rangle$ and let $K = \mathrm{Ker}\psi|_G$. Let $p_1, p_2\colon \mathrm{PGL}_2(\mathbb{C})^2 \to \mathrm{PGL}_2(\mathbb{C})$ be the projections onto the first and second factors, respectively. By Goursat's lemma, we have $K = H_1 \times_Q H_2$, where $H_i = p_i(K)$; in particular, every element of K is of the form $(x,y) \mapsto (h_1x, h_2y)$, where (x,y) are local coordinates on S, $h_1 \in H_1$, $h_2 \in H_2$. The whole group G is the union of two cosets, K and τK, where $\tau(x,y) = (Ay, Bx)$ for some fixed automorphisms $A, B \in \mathrm{Aut}(\mathbb{P}^1)$. Since K is normal in G, then the conjugation by τ sends each automorphism $(h_1,h_2) \in K$ onto (Ah_2A^{-1}, Bh_1B^{-1}). Therefore, H_1 is isomorphic to H_2 via the map $h_1 \mapsto Bh_1B^{-1}$. Indeed, it is clearly an injective homomorphism. To check surjectivity, we use that $\tau^2 = (AB, BA) \in K$, i.e. $AB \in H_1$. So, for any $h_2 \in H_2$ we have that $Ah_2A^{-1} \in H_1$ and hence $(AB)^{-1}Ah_2A^{-1}AB = B^{-1}h_2B \in H_1$ is sent to h_2. □

Notations 3.5.2. *In what follows, the matrices* $\mathrm{R}_n, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$ *and* F *are defined as in Section 1.7.1, and* I *stands for the identity matrix. The elements of* $\mathrm{Aut}(S) \setminus \mathrm{Aut}(S)^\circ$*, i.e. those that do not preserve the rulings of* S *and that act as* $(x,y) \mapsto (\mathrm{M}y, \mathrm{N}x)$ *with* $\mathrm{M}, \mathrm{N} \in \mathrm{PGL}_2(\mathbb{C})$*, will be denoted* $(\mathrm{M}, \mathrm{N}, \sigma)$*. Otherwise, we will write* $(\mathrm{M}, \mathrm{N}, \mathrm{id})$*.*

Every fibre product $G = H_1 \times_Q H_2$, where H_1, H_2 are subgroups of $\mathrm{PGL}_2(\mathbb{C})$, obviously acts on $S = \mathbb{P}^1 \times \mathbb{P}^1$ and forces $\mathrm{Pic}(S)^G \simeq \mathbb{Z}^2$. In contrast, not every group G fitting the exact sequence of the form (11) with $H \subset \mathrm{PGL}_2(\mathbb{C})$ actually embeds into $\mathrm{Aut}(S)$, as the following example shows.

Example 3.5.3. The group $G \simeq \mathbb{Z}_2^5$ obviously fits the exact sequence (11) with $H \simeq \mathrm{V}_4$ and $Q = \{\mathrm{id}\}$. However, it cannot be embedded[4] in $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$. Indeed, by Lemma 3.5.5 below, the normalizer of $H = \langle \mathrm{A}, \mathrm{B}\rangle$ in $\mathrm{PGL}_2(\mathbb{C})$ is $\langle \mathrm{A}, \mathrm{B}\rangle \rtimes \langle \mathrm{C}, \mathrm{D}\rangle \simeq \mathrm{V}_4 \rtimes \mathfrak{S}_3 \simeq \mathfrak{S}_4$. Let $g = (\mathrm{M}, \mathrm{N}, \sigma) \in G$ be an element such that $\psi(g) = \sigma$. Then G is generated by $H \times H$ and g. Thus, by multiplying g by an element of $H \times H$, we may assume that $g = (\mathrm{T}_1, \mathrm{T}_2, \sigma)$, where $\mathrm{T}_1, \mathrm{T}_2 \in \langle \mathrm{C}, \mathrm{D}\rangle$. But $g^2 = (\mathrm{T}_1\mathrm{T}_2, \mathrm{T}_2\mathrm{T}_1, \mathrm{id}) \in H \times H$, which forces $\mathrm{T}_1 = \mathrm{T}_2 \in \{\mathrm{I}, \mathrm{D}, \mathrm{CD}, \mathrm{DC}\}$, or $(\mathrm{T}_1, \mathrm{T}_2) \in \{(\mathrm{C}, \mathrm{C}^2), (\mathrm{C}^2, \mathrm{C})\}$. In all these cases, the group G is isomorphic to $\mathrm{V}_4 \wr \mathbb{Z}_2$.

So, our next goal is to fill the existing gap in the literature: we characterize completely those finite groups which admit a faithful action on $S = \mathbb{P}^1 \times \mathbb{P}^1$.

4. In fact, this group cannot be embedded even into the Cremona group $\mathrm{Cr}_2(\mathbb{C})$, see Beauville (2007).

Theorem 3.5.4. *Let $G \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ be a finite subgroup. Then G is conjugate to a subgroup of one of the following groups:*

GAP	*Order*	*Isomorphism class*	*Generators*
No id	$48n$	$\mathrm{D}_n \times \mathfrak{S}_4$	$(\mathrm{R}_n,\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id}),(\mathrm{I},\mathrm{A},\mathrm{id})$ $(\mathrm{I},\mathrm{B},\mathrm{id}),(\mathrm{I},\mathrm{C},\mathrm{id}),(\mathrm{I},\mathrm{D},\mathrm{id})$
No id	$120n$	$\mathrm{D}_n \times \mathfrak{A}_5$	$(\mathrm{R}_n,\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id})$ $(\mathrm{I},\mathrm{E},\mathrm{id}),(\mathrm{I},\mathrm{F},\mathrm{id})$
$[1440,5848]$	1440	$\mathfrak{S}_4 \times \mathfrak{A}_5$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id}),(\mathrm{C},\mathrm{I},\mathrm{id})$ $(\mathrm{D},\mathrm{I},\mathrm{id}),(\mathrm{I},\mathrm{E},\mathrm{id}),(\mathrm{I},\mathrm{F},\mathrm{id})$
No id	$8n^2$	$\mathrm{D}_n \wr \mathbb{Z}_2$, $n \geqslant 3$	$(\mathrm{R}_n,\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id})$ $(\mathrm{I},\mathrm{I},\sigma)$
$[1152,157849]$	1152	$\mathfrak{S}_4 \wr \mathbb{Z}_2$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id}),(\mathrm{C},\mathrm{I},\mathrm{id})$ $(\mathrm{D},\mathrm{I},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$
No id	7200	$\mathfrak{A}_5 \wr \mathbb{Z}_2$	$(\mathrm{E},\mathrm{I},\mathrm{id}),(\mathrm{F},\mathrm{I},\mathrm{id})$ $(\mathrm{I},\mathrm{I},\sigma)$

The proof of this theorem will be given at the end of the Section. We will begin with a complete description, in terms of matrix generators, of the subgroups of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ that satisfy the exact sequence (11) with H isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$.

Lemma 3.5.5. *Let us consider V_4, $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ as subgroups of $\mathrm{PGL}_2(\mathbb{C})$. Then their normalizers in $\mathrm{PGL}_2(\mathbb{C})$ are the following:*

1. $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C})}(\mathrm{V}_4) \simeq \mathfrak{S}_4$;
2. $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C})}(\mathfrak{A}_4) \simeq \mathfrak{S}_4$;
3. $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C})}(\mathfrak{S}_4) \simeq \mathfrak{S}_4$;
4. $\mathrm{N}_{\mathrm{PGL}_2(\mathbb{C})}(\mathfrak{A}_5) \simeq \mathfrak{A}_5$.

Proof. Recall that all finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are unique up to conjugation. Since the group $\mathfrak{A}_4$ is normal in $\mathfrak{S}_4$, then its normalizer contains $\mathfrak{S}_4$. Similarly, $\mathfrak{S}_4$ contains a normal copy of V_4, hence the normalizer of V_4 in $\mathrm{PGL}_2(\mathbb{C})$ contains $\mathfrak{S}_4$. Further, the normalizer of $\mathfrak{S}_4$ contains $\mathfrak{S}_4$, while the normalizer of $\mathfrak{A}_5$ contains $\mathfrak{A}_5$. Notice that the normalizer of a finite group in $\mathrm{PGL}_2(\mathbb{C})$ is an algebraic group, hence we can use their classification provided by Theorem 1.7.3. In each case, the normalizer is clearly not the whole $\mathrm{PGL}_2(\mathbb{C})$, and not the groups of type (3) or (4), as those are metabelian by Remark 1.7.4 and hence cannot contain a copy of $\mathfrak{S}_4$ or $\mathfrak{A}_5$. The claim follows. □

Next, we show how to conjugate fibred products $H \times_Q H$ to some "standard forms" in $\mathrm{Aut}(S)$. Recall that $\mathrm{Aut}(\mathfrak{A}_5) \simeq \mathfrak{S}_5$ and one has $\mathrm{Aut}(\mathfrak{A}_5)/\mathrm{Inn}\,(\mathfrak{A}_5) \simeq \mathbb{Z}_2$. Any outer automorphism of $\mathfrak{A}_5$ is a conjugation by an odd permutation in $\mathfrak{S}_5$.

Proposition 3.5.6. *Let H be one of the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$.*

1. *Suppose that $H \simeq \mathfrak{A}_4$ or $H \simeq \mathfrak{S}_4$. Then every quotient $Q = H/K$ of H defines uniquely the subgroup K and the fibre product $H \times_Q H$, which is conjugate in $\mathrm{Aut}(S)$ to the group*

$$H \times_Q H = \{(h_1, h_2) \in H \times H \colon \overline{h}_1 = \overline{h}_2\} = \{(h, hk) \colon h \in H, k \in K\}. \qquad (12)$$

In particular, any action of H on S is conjugate to the diagonal one.

2. *Suppose that $H \simeq \mathfrak{A}_5$ and fix an outer automorphism $\xi \in \mathrm{Aut}(\mathfrak{A}_5) \setminus$ Inn $(\mathfrak{A}_5)$. Then any fibre product $H \times_Q H$ is conjugate in $\mathrm{Aut}(S)$ to one of the following groups:*
   (a) $\{(h_1, h_2) \in \mathfrak{A}_5 \times \mathfrak{A}_5\}$;
   (b) $\{(h, h) \colon h \in \mathfrak{A}_5\}$, *i.e. $\mathfrak{A}_5$ acts diagonally on S;*
   (c) $\{(h, \xi(h)) \colon h \in \mathfrak{A}_5\}$.

Proof. Let $K \subseteq H$ be a normal subgroup. The classification of normal subgroups of H implies that the isomorphism type of the quotient $Q = H/K$ determines K uniquely. Vice versa, the isomorphism type of K determines the quotient Q. Therefore, by Goursat's Lemma 1.7.8, the fibred product $H \times_Q H$ is uniquely determined by H, Q and an automorphism $\delta \in \mathrm{Aut}(Q)$, and in this case

$$H \times_Q H = \{(h_1, h_2) \in H \times H \colon \delta(\overline{h}_1) = \overline{h}_2\}$$

(1) Let us show that one can always obtain $\delta = \mathrm{id}$ by conjugating this group in $\mathrm{Aut}(S)$. We may assume that $Q \neq \{\mathrm{id}\}$. Then direct computations show that one has $H = K \rtimes Q$ for some complement Q to K in H; then we identify the quotient Q with a subgroup of H and can write $H = q_1 K \sqcup \ldots \sqcup q_n K$, where $Q = \{q_1, \ldots, q_n\} \subset H$. More precisely, one of the following holds:

(i) $H = \mathfrak{A}_4$, $K = \{\mathrm{id}\}$, $Q = \mathfrak{A}_4$ and $\mathrm{Aut}(Q) \simeq \mathfrak{S}_4$.
(ii) $H = \mathfrak{A}_4$, $K = \mathrm{V}_4$, $Q = \langle (123) \rangle \simeq \mathbb{Z}_3$ and $\mathrm{Aut}(Q) \simeq \mathbb{Z}_2$.
(iii) $H = \mathfrak{S}_4$, $K = \{\mathrm{id}\}$, $Q = \mathfrak{S}_4$ and $\mathrm{Aut}(Q) =$ Inn $(Q) \simeq \mathfrak{S}_4$.
(iv) $H = \mathfrak{S}_4$, $K = \mathrm{V}_4$, $Q = \langle (123), (12) \rangle \simeq \mathfrak{S}_3$ and $\mathrm{Aut}(\mathfrak{S}_3) \simeq$ Inn $(\mathfrak{S}_3) \simeq \mathfrak{S}_3$.
(v) $H = \mathfrak{S}_4$, $K = \mathfrak{A}_4$, $Q = \langle (12) \rangle \simeq \mathbb{Z}_2$.

In particular we see that any automorphism $\delta \in \mathrm{Aut}(Q)$ is a conjugation $g \mapsto hgh^{-1}$ by an element $h \in \mathfrak{S}_4$ (note that in the case (ii) the only non-trivial automorphism $(123) \mapsto (123)^2 = (132)$ is the conjugation by (12)). Since every such h corresponds to an automorphism $\alpha_h \in \mathrm{Aut}(\mathbb{P}^1)$, the automorphism $(h_1, h_2) \mapsto (h_1, \alpha_h^{-1} h_2 \alpha_h)$ conjugates $H \times_Q H$ to the fibre product with $\delta = \mathrm{id}$.

(2) Since $\mathfrak{A}_5$ is simple, then either $Q = \{\mathrm{id}\}$ or $Q = \mathfrak{A}_5$. In the first case, we get the group (a). In the second case, one has $H \times_Q H = \{(h, \delta(h)) \in \mathfrak{A}_5 \times \mathfrak{A}_5\}$ for $\delta \in \mathrm{Aut}(\mathfrak{A}_5)$. If $\delta \in$ Inn $(\mathfrak{A}_5)$, the same argument as in (1) shows that one can conjugate $H \times_Q H$ to the group (b). Otherwise $\delta = \gamma \circ \xi$ for some $\gamma \in$ Inn $(\mathfrak{A}_5)$. Since γ corresponds to the conjugation by an automorphism of $\mathbb{P}^1$, we can conjugate the whole group to (c). □

Proposition 3.5.7. *Assume that $H \simeq \mathfrak{A}_4$. Then, up to conjugation in $\mathrm{Aut}(S)$, there are the following cases for G, and only them.*

GAP	*Order*	*Isomorphism class*	*Generators*
$[24,13]$	24	$\mathfrak{A}_4 \times \mathbb{Z}_2$	$(\mathrm{A},\mathrm{A},\mathrm{id}),(\mathrm{B},\mathrm{B},\mathrm{id})$ $(\mathrm{C},\mathrm{C},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$
$[24,12]$	24	$\mathfrak{S}_4$	$(\mathrm{A},\mathrm{A},\mathrm{id}),(\mathrm{B},\mathrm{B},\mathrm{id})$ $(\mathrm{C},\mathrm{C},\mathrm{id}),(\mathrm{D},\mathrm{D},\sigma)$
$[96,70]$	96	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_6$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id})$ $(\mathrm{C},\mathrm{C},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$
$[96,227]$	96	$\mathbb{Z}_2^2 \rtimes \mathfrak{S}_4$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id})$ $(\mathrm{C},\mathrm{C},\mathrm{id}),(\mathrm{D},\mathrm{D},\sigma)$
$[288,1025]$	288	$\mathfrak{A}_4 \wr \mathbb{Z}_2$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id})$ $(\mathrm{C},\mathrm{I},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$

Proof. Any subgroup of $\mathrm{PGL}_2(\mathbb{C})$ isomorphic to $\mathfrak{A}_4$ is conjugate to the group H generated by the matrices $A,B,$ and C, see Section 1.7.1. We then apply Proposition 3.5.6. Note that $K = \mathrm{V}_4 \triangleleft \mathfrak{A}_4$ is generated by A and B. Then, because of (12), we have one of the following possibilities for $H \times_Q H$:

(1) $K = \{\mathrm{id}\}$ $\langle(\mathrm{A},\mathrm{A}),(\mathrm{B},\mathrm{B}),(\mathrm{C},\mathrm{C})\rangle$;

(2) $K = \mathrm{V}_4$ $\langle(\mathrm{A},\mathrm{I}),(\mathrm{B},\mathrm{I}),(\mathrm{I},\mathrm{A}),(\mathrm{I},\mathrm{B}),(\mathrm{C},\mathrm{C})\rangle$

(3) $K = \mathfrak{A}_4$ $\langle(\mathrm{A},\mathrm{I}),(\mathrm{B},\mathrm{I}),(\mathrm{C},\mathrm{I}),(\mathrm{I},\mathrm{A}),(\mathrm{I},\mathrm{B}),(\mathrm{I},\mathrm{C})\rangle$.

It remains to identify an element $g \in G$ such that $\psi(g) = \sigma$. It is of the form $(\mathrm{M},\mathrm{N},\sigma) \in \mathrm{PGL}_2(\mathbb{C})^2 \rtimes \mathbb{Z}_2$, and normalizes $H \times_Q H$. Since $(\mathrm{M},\mathrm{N},\sigma)^{-1} = (\mathrm{N}^{-1},\mathrm{M}^{-1},\sigma)$, then for any $\mathrm{P} \in H$ we get that

$$(\mathrm{N}^{-1},\mathrm{M}^{-1},\sigma)(\mathrm{P},\mathrm{P},\mathrm{id})(\mathrm{M},\mathrm{N},\sigma) = (\mathrm{N}^{-1}\mathrm{PN},\mathrm{M}^{-1}\mathrm{PM},\mathrm{id}) \tag{13}$$

is an element of $H \times_Q H \subset H \times H$, which implies that M and N normalize H in $\mathrm{PGL}_2(\mathbb{C})$. By Lemma 3.5.5, the normalizer of H in $\mathrm{PGL}_2(\mathbb{C})$ is the group isomorphic to $\mathfrak{S}_4$ which contains H. Consider three cases.

Suppose that $H \times_Q H$ is of the form (1) among the three above possibilities. If $\mathrm{M} \in H$, then, up to a multiplication by an element of $H \times_Q H$, we may assume that $g = (\mathrm{I},\mathrm{N},\sigma)$. Therefore, (13) implies $\mathrm{N}^{-1}\mathrm{PN} = \mathrm{P}$ for all $\mathrm{P} \in H$. But the centre of $\mathfrak{A}_4$ is trivial, hence $\mathrm{N} = \mathrm{I}$. If $M \notin H$, then (13) implies that $\mathrm{N}^{-1}\mathrm{PN} = \mathrm{M}^{-1}\mathrm{PM}$ for all $\mathrm{P} \in H$, hence $\mathrm{MN}^{-1} \in \mathrm{C}_{\mathfrak{S}_4}(\mathfrak{A}_4) = \{\mathrm{id}\}$, and we conclude that $\mathrm{M} = \mathrm{N}$, $g = (\mathrm{M},\mathrm{M},\sigma)$. Note that $\mathrm{M} = \mathrm{TD}$ for $\mathrm{T} \in H$ and D the matrix introduced in Section 1.7.1 (whose coset generates $\mathfrak{S}_4/\mathfrak{A}_4$). Multiplying further g by $(\mathrm{T}^{-1},\mathrm{T}^{-1},\mathrm{id})$, we achieve $g = (\mathrm{D},\mathrm{D}\,\sigma)$.

Now suppose that $H \times_Q H$ is of the form (2). As before, if $\mathrm{M} \in H$, then we may assume $\mathrm{M} = \mathrm{I}$. The condition (13) gives that $(\mathrm{N}^{-1}\mathrm{PN}, \mathrm{P})$ is an element of $H \times_Q H$ for any $\mathrm{P} \in H$, hence $[\mathrm{N}, \mathrm{P}] \in K$ for all $\mathrm{P} \in H$. This implies $\mathrm{N} \in \mathfrak{A}_4$, hence $\mathrm{N} = \mathrm{T}$, $\mathrm{N} = \mathrm{TC}$ or $\mathrm{N} = \mathrm{TC}^2$, where $\mathrm{T} \in \mathrm{V}_4$. Identifying $\langle (\mathrm{I}, \mathrm{A}), (\mathrm{I}, \mathrm{B}) \rangle$ with V_4, we may further multiply g by an element of this group to get $g = (\mathrm{I}, \mathrm{I}, \sigma)$, $g = (\mathrm{I}, \mathrm{C}, \sigma)$ or $g = (\mathrm{I}, \mathrm{C}^2, \sigma)$, respectively. But we can exclude the last two options, because $g = (\mathrm{I}, \mathrm{C}^2, \sigma)$ is the same as $g = (\mathrm{I}, \mathrm{C}, \sigma)$, up to conjugation by $(\mathrm{D}, \mathrm{D}, \mathrm{id})$, which normalizes $H \times_Q H$, while

$$(\mathrm{C}^2, \mathrm{C}, \mathrm{id})^{-1}(\mathrm{I}, \mathrm{C}, \sigma)(\mathrm{C}^2, \mathrm{C}, \mathrm{id}) = (\mathrm{C}, \mathrm{C}, \mathrm{id})^2(\mathrm{I}, \mathrm{I}, \sigma).$$

Now, if $\mathrm{M} \notin H$, then (13) implies that $(\mathrm{N}^{-1}\mathrm{PN}, \mathrm{M}^{-1}\mathrm{PM}, \mathrm{id})$ is an element of $H \times_Q H$ for all $\mathrm{P} \in H$, which is equivalent to $\mathrm{N}^{-1}\mathrm{P}^{-1}\mathrm{NM}^{-1}\mathrm{PM}$ being an element of K for all $\mathrm{P} \in H$. But this holds if and only if $[\mathrm{P}, \mathrm{MN}^{-1}] \in \mathrm{N}K\mathrm{N}^{-1} = K$ for all $\mathrm{P} \in H$. As above, we conclude that $\mathrm{MN}^{-1} \in \mathfrak{A}_4$ and hence $g = (\mathrm{M}, \mathrm{TM}, \sigma)$, where $\mathrm{T} \in \mathfrak{A}_4$. As above, we can multiply g by an element of $\langle (\mathrm{A}, \mathrm{I}), (\mathrm{B}, \mathrm{I}), (\mathrm{I}, \mathrm{A}), (\mathrm{I}, \mathrm{B}) \rangle$ to get $g = (\mathrm{M}, \mathrm{M}, \sigma)$, $g = (\mathrm{M}, \mathrm{CM}, \sigma)$ or $g = (\mathrm{M}, \mathrm{C}^2\mathrm{M}, \sigma)$. Since $\mathrm{M} = \mathrm{DT}'$ for some $\mathrm{T}' \in H$, then with a further multiplication by $(\mathrm{T}'^{-1}, \mathrm{T}'^{-1}, \mathrm{id}) \in H \times_Q H$, we achieve $g \in \{(\mathrm{D}, \mathrm{D}, \sigma), (\mathrm{D}, \mathrm{CD}, \sigma), (\mathrm{D}, \mathrm{C}^2\mathrm{D}, \sigma)\}$. But we can exclude $g = (\mathrm{D}, \mathrm{CD}, \sigma)$ and $g = (\mathrm{D}, \mathrm{C}^2\mathrm{D}, \sigma)$, because the squares of those elements are respectively $(\mathrm{C}^2, \mathrm{C}, \mathrm{id})$ and $(\mathrm{C}, \mathrm{C}^2, \mathrm{id})$, which are not in $H \times_Q H = \langle (\mathrm{A}, \mathrm{I}), (\mathrm{B}, \mathrm{I}), (\mathrm{I}, \mathrm{A}), (\mathrm{I}, \mathrm{B}), (\mathrm{C}, \mathrm{C}) \rangle$.

Finally, assume that $H \times_Q H$ is of type (3). Note that $g^2 = (\mathrm{MN}, \mathrm{NM}, \mathrm{id}) \in H \times H$, therefore $\mathrm{N} = \mathrm{M}^{-1}\mathrm{T}$, where $\mathrm{T} \in \mathfrak{A}_4$. Multiplying g by $(\mathrm{I}, \mathrm{T}^{-1}, \mathrm{id})$, we may assume $g = (\mathrm{M}, \mathrm{M}^{-1}, \sigma)$. Finally, since $\mathrm{M} = \mathrm{T}'$ or $\mathrm{M} = \mathrm{T}'\mathrm{D}$ for $\mathrm{T}' \in \mathfrak{A}_4$, multiplication by an element of $H \times H$ gives $g = (\mathrm{I}, \mathrm{I}, \sigma)$ or $g = (\mathrm{D}, \mathrm{D}, \sigma)$, respectively. But the latter is conjugate to the former in $\mathrm{Aut}(S)$ via the automorphism $(\mathrm{I}, \mathrm{D}, \mathrm{id})$. □

Remark 3.5.8. Some generators of G, obtained during the proofs of Propositions 3.5.7, 3.5.9, may be redundant. This is taken into account in the *Generators* column of the corresponding tables. For example, as soon as G contains $(\mathrm{I}, \mathrm{I}, \sigma)$, the group G also contains $(\mathrm{I}, \mathrm{I}, \sigma)(\mathrm{M}, \mathrm{N}, \mathrm{id})(\mathrm{I}, \mathrm{I}, \sigma) = (\mathrm{N}, \mathrm{M}, \mathrm{id})$ for each $(\mathrm{M}, \mathrm{N}, \mathrm{id}) \in G$. Similarly, suppose that we are in case (2) of the above proof, and $G \simeq \mathbb{Z}_2^2 \rtimes \mathfrak{S}_4$ is generated by $(\mathrm{A}, \mathrm{I}, \mathrm{id})$, $(\mathrm{B}, \mathrm{I}, \mathrm{id})$, $(\mathrm{I}, \mathrm{A}, \mathrm{id})$, $(\mathrm{I}, \mathrm{B}, \mathrm{id})$, $(\mathrm{C}, \mathrm{C}, \mathrm{id})$ and $(\mathrm{D}, \mathrm{D}, \sigma)$. The relations in $\mathfrak{S}_4$ then yield $(\mathrm{I}, \mathrm{B}, \mathrm{id}) = (\mathrm{I}, \mathrm{DBD}, \mathrm{id}) = (\mathrm{D}, \mathrm{D}, \sigma)^{-1}(\mathrm{B}, \mathrm{I}, \mathrm{id})(\mathrm{D}, \mathrm{D}, \sigma)$, and $(\mathrm{I}, \mathrm{A}, \mathrm{id}) = [(\mathrm{D}, \mathrm{D}, \sigma)^{-1}(\mathrm{A}, \mathrm{I}, \mathrm{id})(\mathrm{D}, \mathrm{D}, \sigma)] \cdot (\mathrm{I}, \mathrm{B}, \mathrm{id})^{-1}$. We leave other cases to the reader.

Proposition 3.5.9. *Assume that $H \simeq \mathfrak{S}_4$. Then, up to conjugation in $\mathrm{Aut}(S)$, there are the following cases for G, and only them.*

GAP	*Order*	*Isomorphism class*	*Generators*
$[48,48]$	48	$\mathfrak{S}_4 \times \mathbb{Z}_2$	$(\mathrm{A},\mathrm{A},\mathrm{id}),(\mathrm{B},\mathrm{B},\mathrm{id}),(\mathrm{C},\mathrm{C},\mathrm{id})$ $(\mathrm{D},\mathrm{D},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$
$[192,955]$	192	$\mathbb{Z}_2^4 \rtimes \mathrm{D}_6$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id}),(\mathrm{C},\mathrm{C},\mathrm{id})$ $(\mathrm{D},\mathrm{D},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$
$[576,8654]$	576	$\mathfrak{A}_4^2 \rtimes \mathbb{Z}_2^2$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id}),(\mathrm{C},\mathrm{I},\mathrm{id})$ $(\mathrm{D},\mathrm{D},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$
$[576,8652]$	576	$\mathfrak{A}_4^2 \rtimes \mathbb{Z}_4$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id}),(\mathrm{C},\mathrm{I},\mathrm{id})$ $(\mathrm{D},\mathrm{D},\mathrm{id}),(\mathrm{I},\mathrm{D},\sigma)$
$[1152,157849]$	1152	$\mathfrak{S}_4 \wr \mathbb{Z}_2$	$(\mathrm{A},\mathrm{I},\mathrm{id}),(\mathrm{B},\mathrm{I},\mathrm{id}),(\mathrm{C},\mathrm{I},\mathrm{id})$ $(\mathrm{D},\mathrm{I},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$

Proof. First, we apply Proposition 3.5.6. Proper normal subgroups of $\mathfrak{S}_4$ are $\mathfrak{A}_4$ and V_4, hence the only possibilities for $H \times_Q H$ are the following:

(1) $K = \{\mathrm{id}\}$ $\langle(\mathrm{A},\mathrm{A}),(\mathrm{B},\mathrm{B}),(\mathrm{C},\mathrm{C}),(\mathrm{D},\mathrm{D})\rangle$;
(2) $K = \mathrm{V}_4$ $\langle(\mathrm{A},\mathrm{I}),(\mathrm{B},\mathrm{I}),(\mathrm{I},\mathrm{A}),(\mathrm{I},\mathrm{B}),(\mathrm{C},\mathrm{C}),(\mathrm{D},\mathrm{D})\rangle$;
(3) $K = \mathfrak{A}_4$ $\langle(\mathrm{A},\mathrm{I}),(\mathrm{B},\mathrm{I}),(\mathrm{C},\mathrm{I}),(\mathrm{I},\mathrm{A}),(\mathrm{I},\mathrm{B}),(\mathrm{I},\mathrm{C}),(\mathrm{D},\mathrm{D})\rangle$;
(4) $K = \mathfrak{S}_4$ $\langle(\mathrm{A},\mathrm{I}),(\mathrm{B},\mathrm{I}),(\mathrm{C},\mathrm{I}),(\mathrm{D},\mathrm{I}),(\mathrm{I},\mathrm{A}),(\mathrm{I},\mathrm{B}),(\mathrm{I},\mathrm{C}),(\mathrm{I},\mathrm{D})\rangle$.

We now look for $g = (\mathrm{M},\mathrm{N},\sigma)$ such that G is generated by $H \times_Q H$ and h, i.e. $\psi(g) = \sigma$. By Lemma 3.5.5, the normalizer of H in $\mathrm{PGL}_2(\mathbb{C})$ is H itself. Therefore, $\mathrm{M},\mathrm{N} \in H$. Up to a multiplication by an element of $H \times_Q H$, we may assume that $\mathrm{M} = \mathrm{I}$, so that $g = (\mathrm{I},\mathrm{N},\sigma)$. We again argue case by case.

If $H \times_Q H$ is of type (1), then the normalization condition (13) implies that $\mathrm{N} \in \mathrm{Z}(\mathfrak{S}_4) = \{\mathrm{id}\}$. If $H \times_Q H$ is of type (2), then $[\mathrm{N},\mathrm{P}] \in K$ for all $\mathrm{P} \in H$, which implies $\mathrm{N} \in \mathrm{V}_4$. Multiplying g by an element of $\langle(\mathrm{I},\mathrm{A}),(\mathrm{I},\mathrm{B})\rangle \simeq \mathrm{V}_4$, we get $g = (\mathrm{I},\mathrm{I},\sigma)$. In the case (3), recall that the quotient of $\mathfrak{S}_4$ by $\langle\mathrm{A},\mathrm{B},\mathrm{C}\rangle \simeq \mathfrak{A}_4$ is generated by the coset of D. Therefore, multiplying g by an element of $\langle(\mathrm{I},\mathrm{A}),(\mathrm{I},\mathrm{B}),(\mathrm{I},\mathrm{C})\rangle$, we achieve $g = (\mathrm{I},\mathrm{I},\sigma)$ or $g = (\mathrm{I},\mathrm{D},\sigma)$. Finally, in the case (4), we can always get $g = (\mathrm{I},\mathrm{I},\sigma)$. □

Proposition 3.5.10. *Assume that $H \simeq \mathfrak{A}_5$. Then, up to conjugation in $\mathrm{Aut}(S)$, there are the following cases for G, and only them*[5]*.*

GAP	*Order*	*Isomorphism class*	*Generators*
$[120,35]$	120	$\mathfrak{A}_5 \times \mathbb{Z}_2$	$(\mathrm{E},\mathrm{E},\mathrm{id}),(\mathrm{F},\mathrm{F},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$
$[120,34]$	120	$\mathfrak{S}_5$	$(\mathrm{E},\xi(\mathrm{E}),\mathrm{id}),(\mathrm{F},\xi(\mathrm{F}),\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$
No id	7200	$\mathfrak{A}_5 \wr \mathbb{Z}_2$	$(\mathrm{I},\mathrm{E},\mathrm{id}),(\mathrm{I},\mathrm{F},\mathrm{id}),(\mathrm{E},\mathrm{I},\mathrm{id}),(\mathrm{F},\mathrm{I},\mathrm{id}),(\mathrm{I},\mathrm{I},\sigma)$

Proof. We again let $g = (\mathrm{M},\mathrm{N},\sigma) \in G$ be an element such that $\psi(g) = \sigma$. Since g normalizes $H \times_Q H$, we deduce that M and N normalize H in $\mathrm{PGL}_2(\mathbb{C})$, hence $\mathrm{M},\mathrm{N} \in H$ by Lemma 3.5.5. We now apply Proposition 3.5.6(2). If $H \times_Q H$ is of type (a), then one can multiply g by an element of $H \times_Q H$ to get $g = (\mathrm{I},\mathrm{I},\sigma)$. In case (b), the normalization condition (13) implies $\mathrm{MN}^{-1} \in \mathrm{C}_{\mathfrak{A}_5}(\mathfrak{A}_5) = \mathrm{id}$, hence $\mathrm{M} = \mathrm{N}$ and we can again replace $g = (\mathrm{M},\mathrm{M},\mathrm{id})$ by $(\mathrm{I},\mathrm{I},\sigma)$. Finally, assume that we are in case (c), that is, $H \times_Q H = \{(\mathrm{P},\xi(\mathrm{P}))\colon \mathrm{P} \in H)\}$. Multiplying g by $(\mathrm{M}^{-1},\xi(\mathrm{M}^{-1}),\mathrm{id})$, we may assume $\mathrm{M} = \mathrm{I}$. Since for each $\mathrm{P} \in H$, one has

$$(\mathrm{I},\mathrm{N},\sigma)^{-1}(\mathrm{P},\xi(\mathrm{P}),\mathrm{id})(\mathrm{I},\mathrm{N},\sigma) = (\mathrm{N}^{-1}\xi(\mathrm{P})\mathrm{N}^{-1},\mathrm{P},\mathrm{id}) \in H \times_Q H,$$

we get $\mathrm{P} = \xi(\mathrm{N}^{-1}\xi(\mathrm{P})\mathrm{N})$. Since ξ is of order 2, we get that $\mathrm{N} \in \mathrm{Z}(\mathfrak{A}_5) = \{\mathrm{id}\}$. □

Proof of Theorem 3.5.4. If $\mathrm{Pic}(S)^G \simeq \mathbb{Z}^2$, then G is a subgroup of $H_1 \times H_2$, where H_1 and H_2 are finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ acting fibrewisely on $\mathbb{P}^1 \times \mathbb{P}^1$. Clearly, we can consider the direct products of cyclic and dihedral groups as subgroups of a group of type $\mathrm{D}_n \wr \mathbb{Z}_2$ from the Theorem, for a suitable n. Furthermore, $\mathbb{Z}_n \times \mathfrak{A}_4$ embeds into $\mathrm{D}_n \times \mathfrak{S}_4$, while $\mathbb{Z}_n \times \mathfrak{A}_5$ embeds into $\mathrm{D}_n \times \mathfrak{A}_5$.

Now suppose that $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$ and hence G fits the exact sequence (11). If H is isomorphic to $\mathfrak{A}_4, \mathfrak{S}_4$, or $\mathfrak{A}_5$, then the result follows from Propositions 3.5.7, 3.5.9 and 3.5.10. Let $g = (\mathrm{M},\mathrm{N},\sigma) \in G$ be an element such that $\psi(g) = \sigma$. Then G is generated by $H \times_Q H$ and g. In particular, G is a subgroup of $\widehat{G} = \langle H \times H, g\rangle$.

Assume that H is cyclic. Then, up to conjugation in $\mathrm{PGL}_2(\mathbb{C})$, the group H is generated by R_n. Since g normalizes $H \times_Q H$, we deduce that M and N are diagonal or anti-diagonal. On the other hand, $g^2 \in H \times_Q H$, i.e. the matrices MN and NM are diagonal, hence either M and N are both diagonal, or both anti-diagonal. Let (u,v) be the affine coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, and let σ_0 be the automorphism $(u,v) \mapsto (v,u)$, i.e. $(\mathrm{I},\mathrm{I},\sigma)$. Then g is either of the form $g\colon (u,v) \mapsto (av,bu)$, or of the form $g\colon (u,v) \mapsto (av^{-1},bu^{-1})$ for some $a,b \in \mathbb{C}^*$.

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5. Recall that, according to Proposition 3.5.6, ξ is any fixed outer automorphism of $\mathfrak{A}_5$.

In the first case, the automorphism $\alpha\colon (u,v) \mapsto (u, a^{-1}v)$ commutes with $H \times H$ and conjugates g to $g'\colon (u,v) \mapsto (v, cu)$, where $c = ab$. The group $\alpha^{-1}\widehat{G}\alpha = \langle H \times H, g' \rangle$ contains an isomorphic copy of G. Since the automorphism $g'^2\colon (u,v) \mapsto (cu, cv)$ belongs to $H \times H$, we get $c = \omega_n^k$ for some $k \in \mathbb{Z}$. Multiplying g' by an automorphism $(u,v) \mapsto (u, c^{-1}v)$, which belongs to $H \times H$, we find that $\alpha^{-1}\widehat{G}\alpha$ is generated by $H \times H$ and σ_0; in particular, it is isomorphic to $\mathbb{Z}_n \wr \mathbb{Z}_2$.

In the second case, the automorphism $\beta\colon (u,v) \mapsto (u, v^{-1})$ normalizes $H \times H$ and conjugates g to $g'\colon (u,v) \mapsto (av, b^{-1}u)$. Then $\beta^{-1}\widehat{G}\beta = \langle H \times H, g' \rangle$ contains a copy of G, and one repeats the argument from the previous case.

Assume that $H \simeq \mathrm{D}_n$. Then H is conjugate in $\mathrm{PGL}_2(\mathbb{C})$ to the group generated by R_n and B. We may assume $n \geqslant 3$, since for $H \simeq \mathrm{V}_4$ the group G is conjugate to a subgroup of $\mathfrak{S}_4 \wr \mathbb{Z}_2$. Let $\mathrm{T} \in H$ be any element, then there is $\mathrm{P} \in H$ so that $(\mathrm{T}, \mathrm{P}, \mathrm{id}) \in H \times_Q H$. One has

$$(\mathrm{N}^{-1}, \mathrm{M}^{-1}, \sigma)(\mathrm{T}, \mathrm{P}, \mathrm{id})(\mathrm{M}, \mathrm{N}, \sigma) = (\mathrm{N}^{-1}\mathrm{P}\mathrm{N}, \mathrm{M}^{-1}\mathrm{T}\mathrm{M}, \mathrm{id}) \in H \times H, \tag{14}$$

and hence M normalizes H in $\mathrm{PGL}_2(\mathbb{C})$. Now $C = \langle \mathrm{R}_n \rangle$ is a characteristic subgroup of H for $n \geqslant 3$, hence it is invariant under conjugation by M, and therefore M is diagonal or anti-diagonal. The same obviously holds for N. Now, by multiplying g by an element of $H \times H$, we may assume that M and N are both diagonal or both anti-diagonal. We finish the proof by applying the same process as in the cyclic case. □

3.6 Linearization of finite groups acting on Hirzebruch surfaces

In the final section, we study the linearization of finite groups acting on Hirzebruch surfaces $\mathbb{F}_n$ with $n \geqslant 0$. This section is divided into three parts. In the first part, we study Hirzebruch surfaces $\mathbb{F}_n$ with $n \geqslant 1$. In the second and third parts, we investigate the linearization of groups acting on the quadric surface $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, according to the G-invariant Picard rank.

3.6.1 G-Hirzebruch surfaces

The goal of this section is to prove the following linearization criterion for Hirzebruch surfaces $\mathbb{F}_n$ with $n \geqslant 1$.

Theorem 3.6.1. *Let $n \geqslant 1$ be an integer and $G \subset \mathrm{Aut}(\mathbb{F}_n)$ be a finite group acting on $\mathbb{F}_n$, such that $\pi\colon \mathbb{F}_n \to \mathbb{P}^1$ is a G-conic bundle. Denote by $\widehat{G}$ the image of G in the automorphism group of the base $\mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PGL}_2(\mathbb{C})$. Then G is linearizable if and only if one of the following holds:*

1. *n is odd;*
2. *$\widehat{G}$ is cyclic;*

3. *$\widehat{G}$ is isomorphic to D_m with $m \in \mathbb{Z}$ odd.*

Before proving this, we need some easy lemmas. First, recall the following classical definition.

Definition 3.6.2. Let $n \geqslant 0$ be a non-negative integer.

1. An *elementary transformation* of the Hirzebruch surface $\mathbb{F}_n$ is the following birational transformation. Let $\varphi\colon Y \to \mathbb{F}_n$ be the blow-up of a point p on a fibre F, $\widetilde{F}$ be the strict transform of F, $\widetilde{\Sigma}_n$ be the strict transform of the $(-n)$-section $\Sigma_n \subset \mathbb{F}_n$ and E be the exceptional divisor. We have $(\widetilde{F})^2 = (\varphi^*F - E)^2 = F^2 - 1 = -1$. Then there is a morphism $\psi\colon Y \to Z$ blowing down $\widetilde{F}$. If $p \notin \Sigma_n$, then $\widetilde{\Sigma}_n^2 = \Sigma_n^2 = -n$ and $\widetilde{\Sigma}_n$ intersects $\widetilde{F}$ transversely in exactly one point. Thus, $\psi(\widetilde{\Sigma}_n)^2 = -n+1$ and $Z \simeq \mathbb{F}_{n-1}$. If $p \in \Sigma_n$, then $\widetilde{\Sigma}_n^2 = \Sigma_n^2 - 1 = -n-1$, $\widetilde{\Sigma}_n \cap \widetilde{F} = \varnothing$, so $\psi(\widetilde{\Sigma}_n)^2 = -n-1$ and $Z \simeq \mathbb{F}_{n+1}$. Therefore, one has the following diagram for an elementary transformation of $\mathbb{F}_n$.

$$\begin{array}{ccc} & Y & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \mathbb{F}_n & \dashrightarrow & Z = \mathbb{F}_{n+1} \text{ or } \mathbb{F}_{n-1} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array} \tag{15}$$

2. Similarly, we define a *G-elementary transformation* $\mathbb{F}_n \dashrightarrow \mathbb{F}_m$, where G is a finite group acting on $\mathbb{F}_n$. In the diagram (15), the map φ is the blow-up of a G-orbit of length ℓ, and ψ is the contraction of the strict transforms of the fibres through the blown-up points. We assume that no two points of the orbit lie in the same fibre; in what follows, we refer to this condition as to *"conic bundle general position"*. If the blown-up orbit belongs to Σ_n, then $m = n + \ell$.
3. More generally, given a G-conic bundle $\pi\colon S \to \mathbb{P}^1$, we define a *$G$-elementary transformation* (or a link of type II between G-conic bundles) $\chi\colon S \dashrightarrow S'$, where $\pi'\colon S' \to \mathbb{P}^1$ is another G-conic bundle, as the blow-up of a G-orbit on S, followed by the contraction of the strict transforms of the fibres through the blown-up points. Once again, we assume that no two points of the orbit lie in the same fibre. Note that χ does not change the number of singular fibres, i.e. one has $K_S^2 = K_{S'}^2$.

Lemma 3.6.3. *Any subgroup $G \subset \mathrm{Aut}(\mathbb{F}_n)$ preserves a curve on $\mathbb{F}_n$ which has self-intersection n and is disjoint from Σ_n.*

Proof. Recall that $S = \mathbb{F}_n$ has the unique section $\Sigma_n \subset \mathbb{F}_n$ of self-intersection $-n$. Let F be a fibre of S. The complete linear system $|\Sigma_n + nF|$ induces the birational morphism $\varphi\colon S \to S' \subset \mathbb{P}^{n+1}$, which is the contraction of Σ_n onto the vertex of the cone S'. As Σ_n is preserved by G, this contraction is G-equivariant. Writing $\mathbb{P}^{n+1} = \mathbb{P}(L \oplus V)$ with L being 1-dimensional vector space corresponding to the unique singular (and hence G-fixed) point of S', we see that $\mathbb{P}(V)$ is a G-invariant hyperplane. It intersects S' in a G-invariant rational normal curve of degree n. Its preimage on S under φ is the required curve. □

Now we will analyze G-elementary transformations of Hirzebruch surfaces in greater detail, based on the “arithmetic” of possible G-orbits; a similar idea was used, e.g. in (Cheltsov, 2014, Lemma B.15).

Lemma 3.6.4. *Let $n \geqslant 1$ and consider a G-conic bundle $\pi\colon \mathbb{F}_n \to \mathbb{P}^1$. One has the following:*

1. *Let ℓ be the length of one of the orbits under the action of $\widehat{G}$ on $\mathbb{P}^1$. Then the conic bundle $\mathbb{F}_n$ is G-birational to $\mathbb{F}_{|n+k\ell|}$, for any $k \in \mathbb{Z}$.*
2. *Let ℓ_i, $i = 1, \dots, r$, be the lengths of the orbits under the action of $\widehat{G}$ on $\mathbb{P}^1$, and let $d = \gcd\{\ell_i\}$. Then $\mathbb{F}_n$ is birational to $\mathbb{F}_{|n+kd|}$ for any $k \in \mathbb{Z}$.*

Proof. (1) The blow-up of $\mathbb{F}_n$ at a G-orbit of length ℓ contained in Σ_n and contraction of the proper transforms of the fibres give a G-birational map to $\mathbb{F}_{n+\ell}$. Moreover, the action of G on the $-(n+\ell)$-curve of $\mathbb{F}_{n+\ell}$ is the same as the action of G on the $(-n)$-curve of $\mathbb{F}_n$. It proves the lemma for $k \geqslant 0$. Let C be a G-invariant n-curve, given by Lemma 3.6.3. Notice that since C and Σ_n are sections of the conic bundle, the action of G on C is the same as the action of G on Σ_n. The blow-up of $\mathbb{F}_n$ at a G-orbit of length ℓ contained in C and contraction of the proper transforms of the fibres gives a G-birational map to $\mathbb{F}_{|n-\ell|}$. Moreover, the action of G on the $-|n-\ell|$-curve of $\mathbb{F}_{|n-\ell|}$ is the same as the action of G on the $(-n)$-curve of $\mathbb{F}_n$. We proceed by induction. Suppose that we constructed a sequence of G-elementary transformations $\mathbb{F}_n \dashrightarrow \mathbb{F}_{|n-k\ell|}$, where $k \geqslant 1$. If $n - k\ell \geqslant 0$, then an elementary transformation at a G-orbit of ℓ points, lying on the G-invariant n-curve as above, gives a map $\mathbb{F}_{n-k\ell} \dashrightarrow \mathbb{F}_{|n-(k+1)\ell|}$. If $n - k\ell < 0$ then we use an elementary transformation at a G-orbit lying on $\Sigma_{|n-k\ell|}$ to get $\mathbb{F}_{|n-k\ell|} = \mathbb{F}_{k\ell-n} \dashrightarrow \mathbb{F}_{k\ell-n+\ell} = \mathbb{F}_{|n-(k+1)\ell|}$.

(2) By Bézout’s identity, we have

$$kd = a_1\ell_1 + \ldots + a_s\ell_s - a_{s+1}\ell_{s+1} - \ldots - \ell_r a_r, \tag{16}$$

where $0 \leqslant s \leqslant r$ and $a_i \geqslant 0$ for all $i = 1, \dots, r$. If $k \geqslant 0$, then, by performing G-elementary transformations as in (1), we get a sequence

$$\mathbb{F}_n \overset{\chi_0}{\dashrightarrow} \mathbb{F}_{n+a_1\ell_1} \overset{\chi_1}{\dashrightarrow} \mathbb{F}_{n+a_1\ell_1+a_2\ell_2} \overset{\chi_2}{\dashrightarrow} \ldots \dashrightarrow \mathbb{F}_{n+a_1\ell_1+\ldots+a_s\ell_s} \overset{\chi_s}{\dashrightarrow} \mathbb{F}_{n+a_1\ell_1+\ldots+a_s\ell_s-a_{s+1}\ell_{s+1}} \overset{\chi_{s+1}}{\dashrightarrow} \ldots, \tag{17}$$

which results in $\mathbb{F}_{n+kd}$. Suppose now that $k<0$, and again write kd in the form (16). Perform the sequence of transformations (17) up to the s-th step. The map χ_s then leads to $\mathbb{F}_{|n+a_1\ell_1+\ldots+a_s\ell_s-a_{s+1}\ell_{s+1}|}=\mathbb{F}_{a_{s+1}\ell_{s+1}-n-a_1\ell_1-\ldots-a_s\ell_s}$. We can proceed by mapping

$$\mathbb{F}_{a_{s+1}\ell_{s+1}-n-a_1\ell_1-\ldots-a_s\ell_s} \dashrightarrow \mathbb{F}_{a_{s+2}\ell_{s+2}+a_{s+1}\ell_{s+1}-n-a_1\ell_1-\ldots-a_s\ell_s}=\mathbb{F}_{|n+a_1\ell_1+\ldots+a_s\ell_s-a_{s+1}\ell_{s+1}-a_{s+2}\ell_{s+2}|},$$

and so on, until we get $\mathbb{F}_{|n+kd|}$. □

Now, we deduce several corollaries.

Corollary 3.6.5. *If $\widehat{G}$ is cyclic or isomorphic to D_m with m odd, then $\mathbb{F}_n$ is G-birational to $\mathbb{F}_k$, for any non-negative integer k. In particular, G is linearizable.*

Proof. If $\widehat{G}$ is cyclic, then it fixes a point on $\mathbb{P}^1$. If $\widehat{G}$ is isomorphic to D_m with m odd, then there is an orbit of length 2 and of odd length by Proposition 1.7.6. In both cases, by Lemma 3.6.4 (2), the surface $\mathbb{F}_n$ is G-birational to $\mathbb{F}_{|n+s|}$ for any $s\in\mathbb{Z}$, hence the first claim. In particular, by taking $s=1-n$, we arrive at $\mathbb{F}_1$ where we can G-equivariantly contract the unique (-1)-curve to get $\mathbb{P}^2$. □

Corollary 3.6.6. *The G-conic bundle $\mathbb{F}_n$ is G-birational to any $\mathbb{F}_{|n+2k|}$, for any $k\in\mathbb{Z}$. So, if n is even, then $\mathbb{F}_n$ is G-birational to the G-conic bundle $\mathbb{F}_0\simeq\mathbb{P}^1\times\mathbb{P}^1$. If n is odd, then $\mathbb{F}_n$ is G-birational to $\mathbb{F}_1$ and hence G is linearizable in this case.*

Proof. If $\widehat{G}$ is cyclic, we conclude using Corollary 3.6.5. Otherwise, Proposition 1.7.6 implies that there is always a pair (or a triple) of orbits of $\widehat{G}$ having 2 as the greatest common divisor of their lengths. By Lemma 3.6.4 (2), we conclude that $\mathbb{F}_n$ is G-birational to $\mathbb{F}_{|n+2k|}$, for any $k\in\mathbb{Z}$. □

3.6.2 Quadrics with invariant Picard group of rank 1

We will classify finite linearizable subgroups of $\mathrm{Aut}(S)$ according to the isomorphism class of H in the exact sequence (11).

Proposition 3.6.7. *If H is cyclic, then G is linearizable.*

Proof. We claim that G has a fixed point, so the stereographic projection from it linearizes the action of G. Indeed, the group $H\times_Q H$ is contained in $H\times H$, hence, it has exactly 4 fixed points on S. We can choose the coordinates on S so that these points are

$$p_1=([1:0],[1:0]),\;\; p_2=([1:0],[0:1]),\;\; p_3=([0:1],[1:0]),\;\; p_4=([0:1],[0:1]).$$

Let $g \in G$ be an element which is mapped to the generator of $\mathbb{Z}_2$ in the exact sequence (11), i.e. which does not preserve the rulings of S. Then G is generated by g and $H \times_Q H$. Since g normalizes $H \times_Q H$, it permutes the points p_1, p_2, p_3, p_4. Since g swaps the rulings of S, it fixes p_1 and p_3, or it fixes p_2 and p_4, or it permutes p_1, p_2, p_3 and p_4 cyclically. In the first two cases, we are done. The third case is impossible, since g^2 would swap p_i with p_{i+2}, while it belongs to $H \times_Q H$, i.e. fixes all four points. □

Remark 3.6.8. In the language of Sarkisov program, we linearize the action of G as follows. Let $\eta : T \to S = \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up of a G-fixed point $p \in S$. The surface T is a del Pezzo surface of degree 7 with three exceptional curves F_1, F_2, and E, where E is the η-exceptional divisor, F_1 and F_2 are the preimages of fibres through p. One has $E \cdot F_1 = E \cdot F_2 = 1$, and $F_1 \cdot F_2 = 0$. Then there is a G-contraction $\eta' \colon T \to \mathbb{P}^2$ of F_1 and F_2 onto a pair of points.

We are going to show that G is not linearizable in any of the other cases.

Observation 3.6.9. We use the notation of Section 1.6.1. Every Sarkisov G-link starting from S is either of type I, where η blows up a G-orbit of length 2, or is of type II, and one of the following holds:

1. $S \simeq S'$, $d = d' = 7$, χ is a birational Bertini involution;
2. $S \simeq S'$, $d = d' = 6$, χ is a birational Geiser involution;
3. S' is a del Pezzo surface of degree 5, $d = 5$, $d' = 2$;
4. $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $d = d' = 4$;
5. S' is a del Pezzo surface of degree 6, $d = 3$, $d' = 1$;
6. $S' \simeq \mathbb{P}^2$, $d = 1$, $d' = 2$.

Birational Geiser and Bertini involutions lead to a G-isomorphic surface. By (Yasinsky, 2023, Proposition 4.3), links centred at orbits of length 4 also result in a surface which is G-isomorphic to S. Therefore, any G-link starting from S and leading to a non-isomorphic surface must be centred at an orbit of length $d \in \{1, 2, 3, 5\}$.

Proposition 3.6.10. *Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $G \subset \mathrm{Aut}(S)$ be a finite group such that $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. Assume that in the setting of the exact sequence* (11) *the group H is isomorphic to $\mathfrak{A}_4, \mathfrak{S}_4$ or $\mathfrak{A}_5$. Then S is G-birationally rigid. In particular, G is not linearizable.*

Proof. Note that an orbit of G is a disjoint union of orbits of $H \times_Q H$. Therefore, by Proposition 1.7.6 and Lemma 1.7.11, any orbit of G on S has length $2m \geqslant 4$. Hence S can admit only G-birational Geiser involutions and links at points of degree 4. As was observed before, they give a G-isomorphic surface. □

Proposition 3.6.11. *Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $G \subset \mathrm{Aut}(S)$ be a finite group such that $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. Assume that in the setting of the exact sequence* (11) *the group H is dihedral D_n. Then G is not linearizable.*

Proof. We may assume that there is a G-orbit $\Sigma \subset S$ in general position, i.e. its blow-up gives a del Pezzo surface. Then Lemma 1.7.12 implies that Σ is a disjoint union of at most 2 orbits of $H \times_Q H$. Suppose that $\Sigma = \Sigma_1 \sqcup \Sigma_2$, where $|\Sigma_1| = |\Sigma_2|$. By Proposition 1.7.6 and Lemma 1.7.11, these cardinalities are divisible by 2 or n. Then $|\Sigma|$ is an even number $\geqslant 4$, and therefore S is G-birationally rigid. So, we may assume that Σ is an orbit of $H \times_Q H$. By the same observations, we only need to investigate orbits of size 2, 3 and 5.

Suppose $|\Sigma| = 5$, i.e. $H \simeq \mathrm{D}_5$. The link $S \dashrightarrow S'$ centred at Σ gives a G-del Pezzo surface S' of degree 5. Since Q can be isomorphic to $\{\mathrm{id}\}$, $\mathbb{Z}_2$ or D_5, we find that either G is an extension of $\mathbb{Z}_2$ by D_5, or G contains a copy of $\mathbb{Z}_5^2$ by Lemma 1.7.9. In the latter case, G cannot be embedded into $\mathrm{Aut}(S') \simeq \mathfrak{S}_5$, and neither $\mathrm{D}_5 \times \mathbb{Z}_2$ can. Hence we may assume that $G \simeq \mathrm{F}_5$. But in this case, G cannot be linearized by Proposition 3.4.3.

If $|\Sigma| = 3$, then $H \simeq \mathfrak{S}_3$, and the blow-up of Σ gives a del Pezzo surface T of degree 5. Since Q can be isomorphic to $\{\mathrm{id}\}$, $\mathbb{Z}_2$ or $\mathfrak{S}_3$, we find that either $G \simeq \mathfrak{S}_3 \times \mathbb{Z}_2$, or G contains a copy of $\mathbb{Z}_3^2$ by Lemma 1.7.9. In the latter case, G does not embed into $\mathrm{Aut}(T) \simeq \mathfrak{S}_5$, hence we may assume that we are in the former case. Then our Sarkisov link ends up on a G-del Pezzo surface of degree 6. Since $G \simeq \mathfrak{S}_3 \times \mathbb{Z}_2$, it cannot be linearized by Proposition 3.4.14.

Finally, case $|\Sigma| = 2$ corresponds to a link of type I leading to the del Pezzo surface T of degree 6 with a G-conic bundle structure. All Sarkisov G-links starting from a G-conic bundle with 2 singular fibres are either G-elementary transformations $T \dashrightarrow T'$, or links $T' \to S'$ of type III leading back to a G-del Pezzo surface S' of degree 8, so the composition of such links looks like in Diagram 1. In view of Propositions 3.6.7, 3.6.10 and the previous cases, it is enough to show that G does not fix a point on S', and hence cannot be further linearized. Let us denote cyclically the six (-1)-curves forming a hexagon on T by $E_1, \ldots, E_6$. We may assume that E_1 and E_4 are the exceptional divisors over Σ, so in particular the action of G must swap E_1 and E_4. They both form sections of the G-conic bundle $\pi\colon T \to \mathbb{P}^1$. On the other hand, $E_2 + E_3$ and $E_5 + E_6$ are the two singular fibres of π, whose irreducible components are swapped by G. By Theorem 3.3.1, the G-conic bundle $\pi'\colon T' \to \mathbb{P}^1$ is again a del Pezzo surface of degree 6. Denote by D_1 and D_2 the singular fibres of π'. Besides their irreducible components, the surface T' has two more (-1)-curves L_1 and L_2, which are disjoint sections of π', swapped by G. Note that the points $E_2 \cap E_3$ and $E_5 \cap E_6$ are unique G-fixed points on T. Since a sequence of G-elementary transformations $T \dashrightarrow T'$ is an isomorphism away from G-orbit of curves of length > 1, the singular points of D_1 and D_2 are the unique G-fixed points on T'. Thus, the contraction $T' \to S'$ of L_1 and L_2 leaves no G-fixed points on S'. $\square$

3.6.3 Quadrics with invariant Picard group of rank 2

Suppose that $\mathrm{Pic}(S)^G \simeq \mathbb{Z}^2$. We endow $S = \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $([x_0 : x_1], [y_0 : y_1])$ and fix two projections π_1 and π_2 to $[x_0 : x_1]$ and $[y_0 : y_1]$, respectively. We often use the affine coordinates $x = x_1/x_0$ and $y = y_1/y_0$ on S, so that the points $([1 : 0], [1 : 0])$ and $([0 : 1], [0 : 1])$ correspond to $(0,0)$ and (∞, ∞). We follow Notations 1.7.5 for the generators of finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$.

By Proposition 3.5.1, the group G is the fibred product $H_1 \times_Q H_2$, where H_1 and H_2 are finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$, acting on S fibrewisely (projections of G induced by π_1 and π_2, respectively). The crucial observation, which follows from (Dolgachev & Iskovskikh, 2009, Propositions 7.12, 7.13), is that G is linearizable if and only if there is a sequence of Sarkisov G-links

$$S = \mathbb{F}_0 \overset{\chi_0}{\dashrightarrow} S_1 \overset{\chi_1}{\dashrightarrow} S_2 \overset{\chi_2}{\dashrightarrow} S_3 \overset{\chi_3}{\dashrightarrow} \dots \overset{\chi_{N-1}}{\dashrightarrow} S_N = \mathbb{F}_1 \overset{\chi_N}{\longrightarrow} \mathbb{P}^2,$$

where $S_i \simeq \mathbb{F}_{n_i}$ for all i, the links χ_i for $i = 0, \dots, N-1$ are either G-elementary transformations of Hirzebruch surfaces, or links of type IV (in which case $S_i \simeq S_{i+1} \simeq \mathbb{P}^1 \times \mathbb{P}^1$), and the last link χ_N is necessarily of type III, namely the equivariant blow-down of the unique (-1)-curve on $\mathbb{F}_1$ to the G-fixed point on $\mathbb{P}^2$.

The main result of this subsection is the following.

Theorem 3.6.12. *The group $G = H_1 \times_Q H_2$ is linearizable if and only if one of the following holds:*

1. *Both H_1 and H_2 are cyclic;*
2. *H_1 is cyclic, and $H_2 \simeq \mathrm{D}_n$, where $n \geqslant 1$ is odd, or vice versa;*
3. *$H_1 \simeq \mathrm{D}_n, H_2 \simeq \mathrm{D}_m$, where $n, m \geqslant 3$ are odd, and G is isomorphic to a dihedral group.*

We split the proof into several auxiliary statements.

Lemma 3.6.13. *The group $G = \mathbb{Z}_n \times_Q \mathbb{Z}_m$ is linearizable for any $n, m \in \mathbb{Z}_{>0}$.*

Proof. We may assume that G is a subgroup of $\langle \mathrm{R}_n \rangle \times \langle \mathrm{R}_m \rangle$, and hence fixes the point $([1 : 0], [1 : 0])$; the stereographic projection from this point linearizes G. □

Lemma 3.6.14. *Let n and m be integers, with n odd. The group $G = \mathrm{D}_n \times_Q \mathbb{Z}_m$ is linearizable.*

Proof. The group $\mathbb{Z}_m$ fixes a point on $\mathbb{P}^1$, and hence G acts on the fibre F over this point. Therefore, the image of $H_1 \times_Q H_2$ in $\mathrm{Aut}(F) \simeq \mathrm{PGL}_2(\mathbb{C})$ is isomorphic to D_k with k odd. By Proposition 1.7.6, there is a G-orbit of size k on F. A G-elementary transformation centred at such orbit leads to the Hirzebruch surface $\mathbb{F}_k$ and we are done by Corollary 3.6.6. □

Lemma 3.6.15. *If H_1 or H_2 is isomorphic to $\mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5$, or D_{2n}, then G is not linearizable.*

Proof. We start with the following observation. Consider a G-elementary transformation

$$\begin{array}{ccc} & T & \\ {}^{\eta}\swarrow & & \searrow^{\eta'} \\ \mathbb{F}_{2n} & \overset{\chi}{\dashrightarrow} & \mathbb{F}_m \\ \searrow & & \swarrow \\ & \mathbb{P}^1 & \end{array}$$

centred at a G-orbit $p_1,\dots,p_{2k} \in \mathbb{F}_{2n}$ of length $2k \geqslant 2$. Let us show that m is even. Indeed, assume that m is odd. For $i = 1,\dots,2k$, let $F_i \subset \mathbb{F}_{2n}$ be the fibres through the blown-up points p_i, let $E_i = \eta^{-1}(p_i) \subset T$ be the exceptional divisors over these points, $\widetilde{F_i} \subset T$ be strict transforms of F_i, and $q_i = \eta'(F_i)$ be the resulting G-orbit on $\mathbb{F}_m$. Suppose that the curve $\Sigma_m \subset \mathbb{F}_m$ contains $t \geqslant 0$ points among q_i. Its strict transform $\widetilde{\Sigma}_m \subset T$ under η' then satisfies $\widetilde{\Sigma}_m^2 = -m-t$, it intersects exactly t curves $\widetilde{F_i}$ on T, and $2k-t$ curves E_i on T. The η-images of these $2k-t$ curves are the blown-up points lying on $C = \eta(\widetilde{\Sigma}_m)$. Thus, $C^2 - (2k-t) = \widetilde{\Sigma}_m^2 = -m-t$, hence $C^2 = 2k-2t-m$ is odd, which is impossible[6] on $\mathbb{F}_{2n}$.

Now let H_1 and H_2 be as in the condition of the Lemma. It is sufficient to show that there is no sequence of G-elementary transformations (possibly alternating with links of type IV on $\mathbb{F}_0$) from $S = \mathbb{F}_0$ to $\mathbb{F}_1$. By Proposition 1.7.6 and Lemma 1.7.11, the group $G = H_1 \times_Q H_2$ has only orbits of even length on S. Thus, the first G-elementary transformation brings us to some $\mathbb{F}_{2n}$. Since the lengths of the orbits is preserved under G-fibrewise transformations, a sequence of G-elementary transformations can only lead to Hirzebruch surfaces $\mathbb{F}_N$ with N even. □

It remains to study the linearizability of G when $H_1 \simeq \mathrm{D}_n$ and $H_2 \simeq \mathrm{D}_m$, with n and m odd. By using Notations 1.7.5, we can suppose that $\mathrm{D}_n = \langle \mathrm{R}_n, \mathrm{B}\rangle$ and $\mathrm{D}_m = \langle \mathrm{R}_m, \mathrm{B}\rangle$ as subgroups of $\mathrm{PGL}_2(\mathbb{C})$. Recall that possible fibre products $\mathrm{D}_n \times_Q \mathrm{D}_m$ were essentially described in Lemma 1.7.14.

Lemma 3.6.16. *Let n,m be odd positive integers. If $G = \mathrm{D}_n \times_Q \mathrm{D}_m$ is not isomorphic to a dihedral group, then G is not linearizable.*

Proof. If G is linearizable, then there is a sequence of Sarkisov G-links $S_1 = \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow S_2 \dashrightarrow \dots \dashrightarrow S_k = \mathbb{F}_1 \to \mathbb{P}^2$, where the last map is the G-equivariant blow-down of the unique (-1)-curve on $\mathbb{F}_1$ — a link of type III. The image of this curve is a G-fixed point $p \in \mathbb{P}^2$, and hence G admits a faithful representation in $\mathrm{GL}(T_p\mathbb{P}^2) \simeq \mathrm{GL}_2(\mathbb{C})$. We conclude by Lemma 1.7.14. □

6. Recall that $\mathrm{Pic}(\mathbb{F}_N)$ is generated by the class of a fibre F and the $(-N)$-section Σ_N (or a transversal fibre Σ_0, such that $\Sigma_0 \cdot F = 1$, when $N = 0$). One has $F^2 = 0$, $F \cdot \Sigma_N = 1$, $\Sigma_N^2 = -N$. So, for a curve $C \sim aF + b\Sigma_N$ one has $C^2 = 2ab - Nb$.

Proposition 3.6.17 (Linearization via the Euclidean algorithm)**.** *Let n and m be odd positive integers. Suppose that the group $G = \mathrm{D}_n \times_Q \mathrm{D}_m$ is dihedral. Then G is linearizable.*

Proof. For convenience, we set $\mathrm{D}_1 = \mathbb{Z}_2$, so that $G = \mathrm{D}_n \times_{\mathrm{D}_d} \mathrm{D}_m$, where $d \geqslant 1$ divides both n and m. By Lemma 1.7.9, there is a short exact sequence of groups

$$1 \longrightarrow \langle(\mathrm{R}_n^d, \mathrm{id})\rangle \times \langle(\mathrm{id}, \mathrm{R}_m^d)\rangle \longrightarrow G \xrightarrow{\ \rho\ } \mathrm{D}_d \longrightarrow 1. \qquad (18)$$

Since G is dihedral, then $\ker\rho$ must be cyclic, so $\gcd(n/d, m/d) = 1$ and $\ker\rho$ is generated by the element $(\mathrm{R}_n^d, \mathrm{R}_m^d)$. By Remark 1.7.10, the group G is generated by $(\mathrm{R}_n^d, \mathrm{R}_m^d)$ and $(\mathrm{B}, \mathrm{R}_m^v\mathrm{B})$ when $d = 1$, and by $(\mathrm{R}_n^d, \mathrm{R}_m^d)$, $(\mathrm{R}_n, \mathrm{R}_m^u)$ and $(\mathrm{B}, \mathrm{R}_m^v\mathrm{B})$ when[7] $d > 1$; here, u, v are some positive integers. Let $k = v/2$ if v is even and $k = (v+m)/2$ if v is odd. By conjugating G in $\mathrm{Aut}(S)$ by the automorphism $(\mathrm{id}, \mathrm{R}_m^k)$, we may assume $v = 0$.

Let $M = nm$. Then
$$(\mathrm{R}_n^d, \mathrm{R}_m^d) = (\mathrm{R}_M^{dm}, \mathrm{R}_M^{dn}), \ \ (\mathrm{R}_n, \mathrm{R}_m^u) = (\mathrm{R}_M^m, \mathrm{R}_M^{nu}).$$

Regardless of whether $d = 1$ or $d > 1$, the elements of G are all of the form $(\mathrm{R}_M^a, \mathrm{R}_M^b)$ or $(\mathrm{B}\mathrm{R}_M^a, \mathrm{B}\mathrm{R}_M^b)$ for some positive integers a, b. Since the latter are involutions, we can assume that the characteristic cyclic subgroup of G is generated by an element of the form $(\mathrm{R}_M^a, \mathrm{R}_M^b)$, i.e. by the map $(x,y) \mapsto (\omega_M^a x, \omega_M^b y)$.

Consider the following birational self-map of S and its inverse:

$$\varphi\colon (x,y) \dashrightarrow (x, x^{-1}y), \ \ \varphi^{-1}\colon (x,y) \dashrightarrow (x, xy).$$

Given a biregular automorphism $g\colon (x,y) \mapsto (\alpha x, \beta y)$ of S for some $\alpha, \beta \in \mathbb{C}^*$, we have the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\ g\ } & S \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ S & \xrightarrow{\ \overline{g}\ } & S, \end{array}$$

where $\overline{g}\colon (x,y) \mapsto (\alpha x, \alpha^{-1}\beta y)$ is $\varphi \circ g \circ \varphi^{-1}$. The conjugation of the automorphism (B, B) by φ gives the same action. To sum up, φ is a birational equivalence between $(\mathbb{P}^1 \times \mathbb{P}^1, G)$ and $(\mathbb{P}^1 \times \mathbb{P}^1, \overline{G})$, where $\overline{G}$ is generated by

$$(x,y) \mapsto (\omega_M^a x, \omega_M^{b-a} y), \ \ (x,y) \mapsto (x^{-1}, y^{-1}).$$

---

7. Note that the second case includes the sub-case $n = m = d$. As follows from Goursat's lemma, we have $G \simeq \{(g, \varphi(g))\colon g \in \mathrm{D}_n\}$, where $\varphi \in \mathrm{Aut}(\mathrm{D}_n)$. It is well known that in the standard presentation of D_n, its automorphism group is generated by the maps $r \mapsto r^j$, $s \mapsto r^t s$, which agrees with the generators we chose.

The biregular automorphism $\sigma\colon (x,y)\mapsto(y,x)$ conjugates the map $(x,y)\mapsto(\alpha x,\beta y)$ to $(x,y)\mapsto(\beta x,\alpha y)$ and does not change (B,B). Therefore, up to conjugation by σ, we may assume $0\leqslant a\leqslant b$. We now run the Euclidean algorithm for a and b: first, by iterating the conjugation by φ, we can replace $(\mathrm{R}_M^a,\mathrm{R}_M^b)$ by $(\mathrm{R}_M^a,\mathrm{R}_M^r)$, where r is the remainder of the division of b by a. Using σ, we replace $(\mathrm{R}_M^a,\mathrm{R}_M^r)$ by $(\mathrm{R}_M^r,\mathrm{R}_M^a)$, perform Euclidean division of a by r, and so on. The algorithm results in the generator $(\mathrm{R}_M^\ell,\mathrm{id})$, where $\ell=\gcd(a,b)$.

Hence, we birationally conjugated G to the dihedral group generated by $(\mathrm{R}_M^\ell,\mathrm{id})$ and (B,B). The fibre over $y=1$ is now faithfully acted on by G. Making an elementary transformation at the orbit of size $N=|G|/2$ in this fibre, we arrive to $\mathbb{F}_N$ and hence can further linearize the action of G. □

Proof of Theorem 3.6.12. It follows from Lemmas 3.6.13–3.6.16 and Proposition 3.6.17. □

Appendix: Magma code

We provide the Magma code that determines the GAP ID of finite subgroups of $\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1)\simeq\mathrm{PO}(4)$ defined by their matrix generators. It is also available on the GitHub page of the first author; see Pinardin (2024). The code provides the following functions.

1. $\mathrm{AutPn}(\textbf{points},\textbf{images})$, which, given two tuples of $n+2$ points in $\mathbb{P}^n$ in general position, returns the automorphism of $\mathbb{P}^n$ that sends the first tuple to the second one.
2. $\mathrm{AutQuadSurf}(\mathbf{M}_1,\mathbf{M}_2,\mathbf{s})$. Given an automorphism $f=(\mathbf{M}_1,\mathbf{M}_2,\mathbf{s})\in\mathrm{PGL}_2^2\rtimes\mathbb{Z}_2\cong\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1)$, this function returns the automorphism of $\mathbb{P}^3$ which restricts to f in the automorphism group of the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^1$.
3. $\mathrm{SGPGL}(\textbf{matrices})$, which returns the subgroup of PGL_n generated by a given set of matrices.
4. The main function, $\mathrm{SGAutP1P1}(\textbf{auts})$, which returns the subgroup of PGL_4 inducing the subgroup of automorphisms of $\mathbb{P}^1\times\mathbb{P}^1$ generated by the list of automorphisms **auts**.

The function $\mathrm{IdentifyGroup}(\mathbf{G})$ is a built-in Magma command which returns the GAP ID of a finite group, if it exists. Below, we use the functions mentioned above to check the generators announced in Notation 1.7.5, and the GAP ID's of the groups in Proposition 3.5.7 and Proposition 3.5.9. The last group of this proposition is of order 1152, so does not have a GAP ID. But it is easy to show that its isomorphism class is $\mathfrak{S}_4\wr\mathbb{Z}_2$.

```
K:=AlgebraicClosure(Rationals());
K4:=CartesianPower(K,4);
R<x>:=PolynomialRing(K);
i:=Roots(x^2+1)[1,1];
w5:=Roots(x^5-1)[2,1];

```

```
A:=Matrix(K,2,[1,0,0,-1]);
B:=Matrix(K,2,[0,1,1,0]);
C:=Matrix(K,2,[i,-i,1,1]);
D:=Matrix(K,2,[1,-i,i,-1]);
E:=Matrix(K,2,[1,1-w5-w5^-1,1,-1]);
F:=Matrix(K,2,[w5,0,0,1]);
I:=Matrix(K,2,[1,0,0,1]);


AutPn:=function(points,images)
    points:=[<t:t in P>:P in points];
    images:=[<t:t in Q>:Q in images];
    n:=#points[1]-1;
    A1:=AffineSpace(K,n+1);
    DefPols1:=[&+([A1.j*points[j+1][i]:j in [1..n+1]] cat [-
        points[1][i]]):i in [1..n+1]];
    X1:=Scheme(A1,DefPols1);
    sols1:=RationalPoints(X1);
    eltsM1:=[];
    for i in [1..n+1] do
        P:=[points[i+1,j]:j in [1..n+1]];
        eltsM1:=eltsM1 cat ElementToSequence(sols1[1][i]*Vector(K
            ,n+1,P));
    end for;
    M1:=Transpose(Matrix(K,n+1,eltsM1));
    DefPols2:=[&+([A1.j*images[j+1][i]:j in [1..n+1]] cat [-
        images[1][i]]):i in [1..n+1]];
    X2:=Scheme(A1,DefPols2);
    sols2:=RationalPoints(X2);
    eltsM2:=[];
    for i in [1..n+1] do
        Q:=[images[i+1,j]:j in [1..n+1]];
        eltsM2:=eltsM2 cat ElementToSequence(sols2[1][i]*Vector(K
            ,n+1,Q));
    end for;
    M2:=Transpose(Matrix(K,n+1,eltsM2));
    M:=M2*M1^-1;
    return M;
end function;

AutQuadSurf:=function(M1,M2,s)
    M1:=Matrix(K,2,ElementToSequence(M1));
    M2:=Matrix(K,2,ElementToSequence(M2));
    if s eq 0 then
```

```
        g:=map<K4->K4|x:-><(ElementToSequence(M1*Matrix(K,2,1,[x[1],x[2]])) cat ElementToSequence(M2*Matrix(K,2,1,[x[3],x[4]])))[1],(ElementToSequence(M1*Matrix(K,2,1,[x[1],x[2]])) cat ElementToSequence(M2*Matrix(K,2,1,[x[3],x[4]])))[2],(ElementToSequence(M1*Matrix(K,2,1,[x[1],x[2]])) cat ElementToSequence(M2*Matrix(K,2,1,[x[3],x[4]])))[3],(ElementToSequence(M1*Matrix(K,2,1,[x[1],x[2]])) cat ElementToSequence(M2*Matrix(K,2,1,[x[3],x[4]])))[4]>>;
    else
        g:=map<K4->K4|x:-><(ElementToSequence(M1*Matrix(K,2,1,[x[3],x[4]])) cat ElementToSequence(M2*Matrix(K,2,1,[x[1],x[2]])))[1],(ElementToSequence(M1*Matrix(K,2,1,[x[3],x[4]])) cat ElementToSequence(M2*Matrix(K,2,1,[x[1],x[2]])))[2],(ElementToSequence(M1*Matrix(K,2,1,[x[3],x[4]])) cat ElementToSequence(M2*Matrix(K,2,1,[x[1],x[2]])))[3],(ElementToSequence(M1*Matrix(K,2,1,[x[3],x[4]])) cat ElementToSequence(M2*Matrix(K,2,1,[x[1],x[2]])))[4]>>;
    end if;
    Phi:=map<K4->K4|x:-><x[1]*x[3],x[1]*x[4],x[2]*x[3],x[2]*x[4]>>;
    PhiInv:=map<K4->K4|x:-><x[1]+x[2],x[3]+x[4],x[1]+x[3],x[2]+x[4]>>;
    g1:=map<K4->K4|x:->Phi(g(PhiInv(x)))>;

    points:=[<2,8,1,4>,<1,1,0,0>,<1,0,1,0>,<0,1,0,1>,<0,0,2,1>];
    images:=[g1(points[1]),g1(points[2]),g1(points[3]),g1(points[4]),g1(points[5])];
    return AutPn(points,images);
end function;

SGPGL:=function(matrices)
    dimension:=Nrows(matrices[1]);
    matrices:=[M/Roots(x^dimension-R!Determinant(M))[1,1]:M in matrices];
    G:=sub<GL(dimension,K)|[GL(dimension,K)|M: M in matrices]>;
    D:=[M: M in Center(G) | IsScalar(M)];
    GP:=quo<G|D>;
    return(GP);
end function;

SGAutP1P1:=function(triples)
    matrices:=[AutQuadSurf(t[1],t[2],t[3]):t in triples];
```

```
    return SGPGL(matrices);
end function;

print "GAP ID of the subgroup of PGL_2 generated by A,B and C:", IdentifyGroup(SGPGL([A,B,C]));
print "GAP ID of the subgroup of PGL_2 generated by A,B,C and D:", IdentifyGroup(SGPGL([A,B,C,D]));
print "GAP ID of the subgroup of PGL_2 generated by E and F:", IdentifyGroup(SGPGL([E,F])),"\n";

print "GAP IDs of the subgroups of Aut(P^1xP^1) given in Proposition 5.12:\n";

IdentifyGroup(SGAutP1P1([<A,A,0>,<B,B,0>,<C,C,0>,<I,I,1>]));
IdentifyGroup(SGAutP1P1([<A,A,0>,<B,B,0>,<C,C,0>,<D,D,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,C,0>,<I,I,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,C,0>,<I,C,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,C,0>,<I,C^2,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,C,0>,<D,D,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,C,0>,<D,C*D,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,C,0>,<D,C^2*D,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,I,0>,<I,I,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,I,0>,<D,D,1>]));

print "\n GAP IDs of the subgroups of Aut(P^1xP^1) given in Proposition 5.12:\n";

IdentifyGroup(SGAutP1P1([<A,A,0>,<B,B,0>,<C,C,0>,<D,D,0>,<I,I,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,C,0>,<D,D,0>,<I,I,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,I,0>,<D,D,0>,<I,I,1>]));
IdentifyGroup(SGAutP1P1([<A,I,0>,<B,I,0>,<C,I,0>,<D,D,0>,<I,D,1>]));

//The last group is of order 1152, hence does not have a GAP ID. But it is easy to show that its isomorphism class is S4 \wr C_2.

G:=SGAutP1P1([<A,I,0>,<B,I,0>,<C,I,0>,<D,I,0>,<I,I,1>]);
print "Last group. Order: ",Order(G)," description: ",GroupName(G);
```

Chapter 4

$\mathfrak{A}_5$-equivariant geometry of quadric threefolds

"Cela ne sert a rien, sauf si je dois compter jusqu'a cinq. Mais il ne faut pas que je dépasse cinq, car je ne connais pas six. Je suis battu."

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We classify G-Mori fibre spaces equivariantly birational to smooth quadric threefolds with fixed-point free actions of the alternating group $G = \mathfrak{A}_5$. We deduce that such quadric threefolds are G-solid and the G-actions on them are not linearizable. The results presented in this chapter have been obtained in collaboration with Zhijia Zhang, see Pinardin and Zhang (2025b). All authors have approved the inclusion of this work in the present thesis and acknowledge equal contribution.

4.1 Introduction

Throughout, the notation G will stand for the alternating group $\mathfrak{A}_5$ of order 60, unless otherwise specified. We restrict ourselves to smooth quadric threefolds $X \subset \mathbb{P}^4$ carrying generically free actions of G such that $X^G \neq \emptyset$, since otherwise a projection from a G-fixed point on X yields a G-birational map $X \dashrightarrow \mathbb{P}^3$. The arising G-actions on $\mathbb{P}^3$ also have fixed points.

Over a non-algebraically closed field, a smooth quadric hypersurface is rational if and only if it has a rational point. Surprisingly, linearizability of group actions on quadrics is a more intricate problem, see, e.g., Hassett and Tschinkel (2024). Many obstructions naturally vanish on quadrics, including group cohomology, see Bogomolov and Prokhorov (2013); Kresch and Tschinkel (2022a) and the dual complex of Esser (2024). See (Cheltsov, Tschinkel, & Zhang, 2024, Section 2) for an overview of known obstructions. Non-linearizable actions on quadric threefolds have been found using the Burnside formalism in (Tschinkel et al., 2023, Example 9.2) and Noether-Fano inequality in Cheltsov, Sarikyan, and Zhuang (2023); Cheltsov and Shramov (2014). However, the first is not applicable to our case.

From representation theory, we know that any fixed-point free G-action on a smooth quadric threefold X is isomorphic to one of the following two cases, which we refer to as the *standard* and *nonstandard* actions:

1. *standard action*:

$$X = X_1 = \left\{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0\right\} \subset \mathbb{P}^4_{x_1,\dots,x_5}$$

with the G-action generated by

$$(\mathbf{x}) \mapsto (x_2, x_1, x_4, x_3, x_5), \quad (\mathbf{x}) \mapsto (x_5, x_1, x_2, x_3, x_4). \tag{1}$$

2. *nonstandard action*:

$$X = X_2 = \left\{ \sum_{1 \le i \le j \le 5} x_i x_j = 0 \right\} \subset \mathbb{P}^4_{x_1,\dots,x_5} \tag{2}$$

with the G-action generated by

$$\begin{aligned}(\mathbf{x}) &\mapsto (x_4, x_1, x_5, x_2, -x_1 - x_2 - x_3 - x_4 - x_5),\\ (\mathbf{x}) &\mapsto (x_4, -x_1 - x_2 - x_3 - x_4 - x_5, x_1, x_3, x_2).\end{aligned} \tag{3}$$

Our goal is to find all G-Mori fibre spaces that are G-birational to X_1 and X_2 respectively, using classical techniques from birational rigidity, based on the celebrated Noether–Fano inequalities. The same has been carried out in Cheltsov, Sarikyan, and Zhuang (2023) to show the non-linearizability of the $\mathfrak{S}_5$-action on X_1 via the $\mathfrak{S}_5$-permutations on coordinates. Our work generalizes their arguments to other actions.

These quadrics are G-birational to certain singular cubic threefolds. By (Cheltsov, Tschinkel, & Zhang, 2024, Section 6), up to isomorphism, there exists a unique cubic threefold Y_1 with $5\mathsf{A}_1$-singularities and invariant under the G-action given by (1). By (Cheltsov, Marquand, et al., 2024, Lemma 8.3), there is a unique cubic threefold Y_2 with $5\mathsf{A}_2$-singularities and invariant under the G-action given by (3). See Section 4.2 for explicit equations of Y_1 and Y_2.

Our main results are the following:

Theorem 4.1.1. *The only G-Mori fibre spaces that are G-birational to the quadric threefold X_1 are X_1 and the cubic threefold Y_1.*

Theorem 4.1.2. *The only G-Mori fibre spaces that are G-birational to the quadric threefold X_2 are X_2 and the cubic threefold Y_2.*

There is also a nonstandard $G' = \mathfrak{S}_5$-action on X_2, generated by (3) and the involution

$$(\mathbf{x}) \mapsto (x_3, x_4, x_1, x_2, -x_1 - x_2 - x_3 - x_4 - x_5).$$

We prove:

Theorem 4.1.3. *The only G'-Mori fibre spaces that are G'-equivariantly birational to the quadric threefold X_2 are X_2 and the cubic threefold Y_2.*

A G-variety is called G-solid if it is not G-birational to a G-Mori fibre space over a positive dimensional base. Our results together with those of (Cheltsov, Sarikyan, & Zhuang, 2023, Theorem 3.1) imply that:

Corollary 4.1.4. *Let $G = \mathfrak{A}_5$ or $\mathfrak{S}_5$, and X a smooth quadric threefold carrying a generically free G-action. Then the following are equivalent*

- *G does not fix any point on X,*
- *the G-action on X is not linearizable,*
- *X is G-solid.*

Note that all such actions on quadric threefolds are known to be stably linearizable by (Cheltsov, Tschinkel, & Zhang, 2025, Theorem 4.1).

Here's the roadmap of the paper: in Section 1.6.4, we recall basic tools from birational geometry. In Section 4.2, we present facts about $\mathfrak{A}_5$-equivariant geometry of quadrics. In Sections 4.3 – 4.6, we prove technical results on singularities of certain log pairs on quadric and cubic threefolds. In Section 4.7, we prove that these technical results imply Theorem 4.1.1 and Theorem 4.1.2, and derive a proof of Theorem 4.1.3.

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4.2 $\mathfrak{A}_5$-actions on quadric threefolds

Let X be a smooth quadric threefold carrying a generically free regular action of $G = \mathfrak{A}_5$. Assume that there exists a G-orbit Σ of 5 points in general position in X. Up to a change of variables, we may also assume that the five points are five coordinate points of $\mathbb{P}^4$. Consider the standard Cremona transformation on $\mathbb{P}^4$

$$\chi\colon (x_1, x_2, x_3, x_4, x_5) \mapsto (\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}, \frac{1}{x_5}).$$

The restriction of χ to X is a G-birational map. The image $\chi(X)$ is a singular cubic threefold. We say that χ is the Cremona map associated with Σ. More descriptions of χ can be found in Avilov (2016b, 2018); Cheltsov, Sarikyan, and Zhuang (2023).

Assume that $X^G \neq \emptyset$. From representation theory, there are two possibilities for the G-action on the ambient $\mathbb{P}^4$:

- *the standard action*: $\mathbb{P}^4 = \mathbb{P}(\mathbf{1} \oplus V_4)$, where V_4 is the unique irreducible 4-dimensional representation of G,
- *the nonstandard action*: $\mathbb{P}^4 = \mathbb{P}(V_5)$, where V_5 is the unique irreducible 5-dimensional representation of G.

4.2.1 The standard action

Under the standard G-action on $\mathbb{P}^4$, up to change of variables, we may assume that X is given by

$$\left\{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0\right\} \subset \mathbb{P}^4_{x_1,\dots,x_5}$$

and the G-action is given by $\mathfrak{A}_5$-permutations of 5 coordinates. There are two G-orbits of length 5, denoted by Σ_5 and Σ_5'. Let

$$Y_1 = \chi_1(X), \quad Y_2 = \chi_1'(X)$$

where χ_1 and χ_1' are the Cremona maps associated with Σ_5 and Σ_5'. One can check by direct computation that Y_1 and Y_1' are cubic threefolds with $5\mathsf{A}_1$-singularities. By (Cheltsov, Tschinkel, & Zhang, 2024, Section 6), such cubics with $\mathfrak{A}_5$-actions are unique up to isomorphism. In particular, we may assume that $Y_1 = Y_1' = Y$ where $Y \subset \mathbb{P}^4$ is given by

$$\{x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + \\ + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 = 0\} \subset \mathbb{P}^4_{x_1,\dots,x_5}$$

and the G-action is still given by permutations of coordinates.

4.2.2 The nonstandard action of $\mathfrak{A}_5$

Up to isomorphism, we may assume that the G-action is as in (3). There is a unique G-invariant quadric $X \subset \mathbb{P}^4$, and it is given by the equation (2). There are also two G-orbits of length 5 in X. Let χ_2 and χ_2' be the birational maps associated with them respectively, and

$$Y_2 = \chi_2(X), \quad Y_2' = \chi_2'(X).$$

One can check that Y_2 and Y_2' are cubic threefolds with $5\mathrm{A}_2$-singularities. Such cubic threefolds with $\mathfrak{A}_5$-actions are unique up to isomorphism by (Cheltsov, Marquand, et al., 2024, Lemma 8.3). Thus, we may assume that $Y = Y_2 = Y_2'$ where Y is given by

$$Y = \{(8-3\zeta_6)f_1 + 7f_2 = 0\} \subset \mathbb{P}^4,$$

for

$$f_1 = x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 + x_2^2x_3 + x_2x_3^2 + 2x_2x_3x_4 + x_3^2x_4 + x_3x_4^2 + \\ + x_1^2x_5 + 2x_1x_2x_5 + 2x_1x_4x_5 + 2x_3x_4x_5 + x_4^2x_5 + x_1x_5^2 + x_4x_5^2,$$

$$f_2 = x_1^2x_3 + x_1x_3^2 + x_1^2x_4 + 2x_1x_2x_4 + x_2^2x_4 + 2x_1x_3x_4 + x_1x_4^2 + x_2x_4^2 + \\ + x_2^2x_5 + 2x_1x_3x_5 + 2x_2x_3x_5 + x_3^2x_5 + 2x_2x_4x_5 + x_2x_5^2 + x_3x_5^2,$$

with the same G-action given by (3).

4.3 The standard $\mathfrak{A}_5$-action on the quadric threefold

Throughout this section, X is the quadric given by

$$\left\{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0\right\} \subset \mathbb{P}^4_{x_1,\ldots,x_5}.$$

Consider the G-action on X given by natural $\mathfrak{A}_5$-permutations of the coordinates. We denote by Σ_5 and Σ_5' two G-orbits of length five on X. The aim of this section is to prove the following proposition.

Proposition 4.3.1. *Let $\mathscr{M}_X$ be a non-empty mobile G-invariant linear system on X, and $\lambda \in \mathbb{Q}$ such that $\lambda\mathscr{M}_X \sim_{\mathbb{Q}} -K_X$. Then the log pair $(X, \lambda\mathscr{M}_X)$ is canonical away from $\Sigma_5 \cup \Sigma_5'$.*

Proof. This follows from Propositions 4.3.10 and 4.3.13, and Corollary 4.3.15.

□

First, as a guiding principle, we observe that curves of degrees greater than 17 cannot be non-canonical centers of $(X, \lambda\mathscr{M}_X)$.

Remark 4.3.2. If a curve C is a center of non-canonical singularities of $(X, \lambda\mathscr{M}_X)$, then for two general members $M_1, M_2 \in \mathscr{M}_X$, we have that

$$\lambda^2(M_1 \cdot M_2) = mC + \Delta$$

for some $m > 1$ and some effective divisor Δ not supported along C. Intersecting with a general hyperplane H on X, we obtain that

$$18 = \lambda^2(M_1 \cdot M_2 \cdot H) > \deg(C). \tag{4}$$

Later, we will see that the size of 0-dimensional non-canonical centers is less than 20, using Nadel vanishing theorem.

We proceed with subsections. In the first subsection, we classify orbits of length less than 20 and G-irreducible curves of degrees at most 17. In the second subsection, we prove that a G-invariant curve not contained in Q cannot be a non-canonical center of $(X, \lambda\mathscr{M}_X)$, where Q is the unique G-invariant hyperplane section on X (cf. Proposition 4.3.10). In the third subsection, we show that points away from Q and $\Sigma_5 \cup \Sigma_5'$ cannot be non-canonical centers (cf. Proposition 4.3.13). In the fourth subsection, using the G-equivariant α-invariant, we prove that no point or curve in Q is a non-canonical center (cf. Corollary 4.3.15).

4.3.1 Small G-orbits and G-invariant curves of low degrees

Lemma 4.3.3. *A G-orbit of points in X with length < 20 is one of the following:*

$$\begin{aligned}
\Sigma_5 &= \textit{the orbit of} \quad [1:1:1:2\zeta_4:1],\\
\Sigma_5' &= \textit{the orbit of} \quad [1:1:1:-2\zeta_4:1],\\
\Sigma_{10} &= \textit{the orbit of} \quad [1:1:\frac{\zeta_4\sqrt{6}}{2}:\frac{\zeta_4\sqrt{6}}{2}:1],\\
\Sigma_{10}' &= \textit{the orbit of} \quad [1:1:-\frac{\zeta_4\sqrt{6}}{2}:-\frac{\zeta_4\sqrt{6}}{2}:1],\\
\Sigma_{12} &= \textit{the orbit of} \quad [1:\zeta_5:\zeta_5^2:\zeta_5^3:\zeta_5^4],\\
\Sigma_{12}' &= \textit{the orbit of} \quad [1:\zeta_5^2:\zeta_5^4:\zeta_5:\zeta_5^3],
\end{aligned}$$

where the length of each orbit is indicated by the subscript.

Proof. This comes from a computation of fixed points by each subgroup of G. □

Lemma 4.3.4. *Every G-invariant curve C in X with $\deg(C) \leq 17$ has a trivial generic stabilizer, that is, the G-orbit of a general point in C has length 60.*

Proof. By computation, we find that all irreducible curves in X with a non-trivial generic stabilizer are conics whose G-orbits have length 10 or 15, and thus their degrees exceed 17. □

There is a distinguished G-invariant hyperplane section of X given by

$$Q = \{x_1 + x_2 + x_3 + x_4 + x_5 = 0\} \cap X.$$

Note that $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and that G acts on Q via two non-isomorphic G-actions on each copy of $\mathbb{P}^1$. Moreover, we have

$$\Sigma_5, \Sigma_5', \Sigma_{10}, \Sigma_{10}' \notin Q, \quad \Sigma_{12}, \Sigma_{12}' \in Q.$$

Let B_6 be the G-invariant smooth curve of degree 6 given by

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0, \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0. \end{cases} \tag{5}$$

It is known as the Bring curve and has genus 4, see (Cheltsov & Shramov, 2016b, Remark 5.4.2).

Lemma 4.3.5. *Let C be a G-invariant reducible curve in X such that $10 < \deg(C) \leq 17$. Then C is the union of curves in one of the following G-orbits:*

- *one of the following 2 orbits of 6 conics*

$$\mathscr{C}_6 = \textit{orbit of } C_1, \qquad \mathscr{C}_6' = \textit{orbit of } C_2,$$

where

$$C_1 = \{x_1 - x_3 + (-\zeta_{20}^6 + \zeta_{20}^4 + 1)x_4 + (\zeta_{20}^6 - \zeta_{20}^4 - 1)x_5 = \\ = x_2 + (\zeta_{20}^6 - \zeta_{20}^4 - 1)x_3 + (-\zeta_{20}^6 + \zeta_{20}^4 + 1)x_4 - x_5 = 0\} \cap X,$$

$$C_2 = \{x_1 - x_3 + (\zeta_{20}^6 - \zeta_{20}^4)x_4 + (-\zeta_{20}^6 + \zeta_{20}^4)x_5 = \\ = x_2 + (-\zeta_{20}^6 + \zeta_{20}^4)x_3 + (\zeta_{20}^6 - \zeta_{20}^4)x_4 - x_5 = 0\} \cap X.$$

- *one of the following 2 orbits of 12 lines*

$$\mathscr{L}_{12} = \textit{the orbit of the line } \{x_1 + \zeta_5 x_4 + (\zeta_5^3 + \zeta_5 + 1)x_5 = x_2 + \\ + (\zeta_5^3 + 1)x_4 + (\zeta_5^2 + 1)x_5 = x_3 - (\zeta_5^3 + \zeta_5)x_4 + \zeta_5^4 x_5 = 0\},$$

$$\mathscr{L}_{12}' = \textit{the orbit of the line } \{x_1 + \zeta_5^2 x_4 + (\zeta_5^2 + \zeta_5 + 1)x_5 = x_2 + \\ + (\zeta_5 + 1)x_4 - (\zeta_5^3 + \zeta_5^2 + \zeta_5)x_5 = x_3 - (\zeta_5^2 + \zeta_5)x_4 + \zeta_5^3 x_5 = 0\}.$$

Each of the orbits above consists of pairwise disjoint components. The orbits $\mathscr{L}_{12}$ and $\mathscr{L}_{12}'$ are contained in Q. The orbits $\mathscr{C}_6$ and $\mathscr{C}_6'$ are not.

Proof. From indices of strict subgroups of G, we find that $\deg(C) = 12$ or 15. If $\deg(C) = 15$, then C is a union of 5 twisted cubics. Each of the twisted cubic receives a generically free $\mathfrak{A}_4$-action and spans a $\mathbb{P}^3$. The $\mathfrak{A}_4$-action on $\mathbb{P}^3$ should have two invariant lines. We check that this does not happen for the given $\mathfrak{A}_4$-action in our case. So this case is impossible.

If $\deg(C) = 12$, then C is either a union of 6 conics or 12 lines. If C contains a conic, the plane spanned by the conic is left invariant by a subgroup $\mathfrak{D}_5 \subset G$. We check that the unique (up to conjugation) $\mathfrak{D}_5$ in G leaves invariant two planes in $\mathbb{P}^4$, giving rise to $\mathscr{C}_6$ and $\mathscr{C}_6'$. If C consists of 12 lines, each line is left invariant by some subgroup $C_5 \subset G$, and thus contains two C_5-fixed points. Then, a computation of C_5-fixed points leads us to $\mathscr{L}_{12}$ and $\mathscr{L}_{12}'$. □

Lemma 4.3.6. *Let C be a G-invariant curve in X with $\deg(C) \leqslant 10$. Then C is contained in $Q = \mathbb{P}^1 \times \mathbb{P}^1$, and is one of the following:*

- *a smooth irreducible curve of bidegree $(1,7)$ and genus 0,*
- *a smooth irreducible curve of bidegree $(2,6)$ and genus 5,*
- *the Bring curve B_6 of bidegree $(3,3)$ and genus 4,*
- *a smooth irreducible curve of bidegree $(4,4)$ of genus 9,*
- *a union of 5 conics of bidegree $(5,5)$.*

Proof. Assume that C is not contained in Q. Then $Q \cdot C = \deg(C)$ and $Q \cap C$ consists of a G-orbit of points in Q of length $\deg(C)$. From the information of orbits in Lemma 4.3.3, we see that $\deg(C) \geq 12$. Thus, the curve C is contained in Q. A computation of G-invariant divisors in Q of bidegree (r_1, r_2) with $r_1 + r_2 \leq 10$ completes the proof. □

Now, we want to classify the G-invariant irreducible curves of degrees at most 17 which are not contained in Q. The strategy is that for each such curve C, we find a G-invariant K3 surface containing C and use the geometry of the K3 surface to proceed. In particular, we are interested in the pencil $\mathscr{P}$ consisting of G-invariant K3 surfaces on X given by

$$S_{a_1,a_2} := \{a_1 f^3 + a_2 g = 0\} \cap X, \quad [a_1 : a_2] \in \mathbb{P}^1$$

where

$$f = \sum_{i=1}^{5} x_i \quad \text{and} \quad g = \sum_{i=1}^{5} x_i^3.$$

Note that the base locus of $\mathscr{P}$ is the Bring curve B_6. We can find singular members in $\mathscr{P}$ by direct computations.

Lemma 4.3.7. *A surface S_{a_1,a_2} in $\mathscr{P}$ is reduced and singular if and only if one of the following holds:*

- $[a_1 : a_2] = [4 \pm 3\zeta_4 : 50] \in \mathbb{P}^1$. *In these cases,* $\mathrm{Sing}(S_{a_1,a_2})$ *consists of* 5 *nodes.*
- $[a_1 : a_2] = [6 \pm \zeta_4\sqrt{3/2} : 75] \in \mathbb{P}^1$. *In these cases,* $\mathrm{Sing}(S_{a_1,a_2})$ *consists of* 10 *nodes.*

Moreover, when S_{a_1,a_2} is smooth, it does not contain $\Sigma_5, \Sigma'_5, \Sigma_{10}$ or Σ'_{10}.

Remark 4.3.8. The orbits $\mathscr{C}_6$ and $\mathscr{C}'_6$ are contained in $S_{2,25} \in \mathscr{P}$.

Lemma 4.3.9. *Let C be an G-invariant curve not contained in Q such that $\deg(C) \leqslant 17$. Then the following statements hold.*

1. $\deg(C) = 12$.
2. *In the pencil $\mathscr{P}$, there is a unique surface S containing C.*
3. *If C is irreducible, then C is a Cartier divisor on S.*
4. *The surface S is smooth.*
5. *The curve C is smooth.*
6. *If C is irreducible, then its genus $g(C) \in \{0, 5, 10\}$.*
7. *There exists a G-invariant curve C' different from C such that $C' \subset S$, C' is isomorphic to C, and $C + C' \sim_{\mathbb{Q}} \mathscr{O}_S(4)$.*

Proof. We may assume that C is G-irreducible.

1. Arguing as in Lemma 4.3.6, we know that $\deg(C) = 12$.
2. Let P be a general point on C. There exists a unique $S \in \mathscr{P}$ such that $P \in S$. If the curve C is not contained in S, then the number of points in $C \cap S$ is at most $3\deg(C) = 36$. But by Lemma 4.3.4, the G-orbit of P has length 60. By contradiction, we see that $C \subset S$.
3. In what follows, we will denote by H a general hyperplane section on X, and by H_S its restriction to S. If the curve C is contained in the smooth locus of S, then it is Cartier. Assume that $C \cap \mathrm{Sing}(S)$ is not empty. Let $f\colon \widetilde{S} \to S$ be the blowup of $C \cap \mathrm{Sing}(S)$, and $\widetilde{C}$ the strict transform of C by f. We have

$$\widetilde{C} \sim_{\mathbb{Q}} f^*(C) - mE, \quad m \in \frac{1}{2}\mathbb{Z},$$

where E is the exceptional divisor of f. To show that C is Cartier, it suffices to prove that m is an integer. Denoting by E_P the component of E mapped to P, we have $\widetilde{C} \cdot E_P = 2m$. But this intersection number is preserved by the action of the stabilizer of P. By Lemma 4.3.7, we have $s = |C \cap \mathrm{Sing}(S)| \in \{5, 10\}$. If $s = 5$, the stabilizer of P is $\mathfrak{A}_4$, and $2m = 4a + 6b + 12c$, where $a, b, c \in \mathbb{Z}_{\geq 0}$, since $4, 6$, and 12 are the possible lengths of $\mathfrak{A}_4$-orbits on $\mathbb{P}^1$. It follows that m is an integer, and C is Cartier. If $s = 10$, then the stabilizer of P is $\mathfrak{S}_3$, and $2m = 2a + 3b + 6c$. If $b = 0$, then m is an integer and we are done. Assume that $b \geq 1$. We have

$$\widetilde{C}^2 = (f^*(C) - mE)^2 = C^2 - 2sm^2 \leq C^2 - 35.$$

By Hodge index theorem, we have

$$C^2 \leq \frac{(C \cdot H_S)^2}{(H_S)^2} = 24, \quad \text{and} \quad \widetilde{C}^2 \leq -11,$$

which is impossible and we obtain a contradiction.

4. If C is reducible, the assertion follows from Remark 4.3.8. Assume that C is irreducible. From (Cheltsov & Shramov, 2016a, Proposition 6.7.3), we know that $\mathrm{rk}(\mathrm{Pic}^G(S)) =$1 or 2, and S is smooth in the latter case. Assume that S is singular, then $\mathrm{Pic}^G(S) = \mathbb{Z}$ and it is generated by H_S since $(H_S)^2 = 6$ is not a square. It follows that $C \sim nH_S$, for some positive integer n. Note that $\deg(C) = 12$ implies that $n = 2$. But one can check that all G-invariant quadratic forms on $\mathbb{P}^4$ are linear combinations of $\sum_{i=1}^5 x_i^2$ and $(\sum_{i=1}^5 x_i)^2$. We deduce that no G-invariant curve in S is linearly equivalent to $2H_S$, hence we obtain a contradiction.
5. If C is reducible, the assertion follows from Lemma 4.3.5. Assume that C is irreducible and singular, the singular locus of C is a union of G-orbits. Since S is smooth, the curve C does not contain any orbit of length $\leq$10, by Lemma 4.3.3. Hence, $\mathrm{Sing}(C)$ must be an orbit of length at least 12. Let us show that this is impossible. Again, Hodge index theorem gives
$$C^2 \leq \frac{(C \cdot H_S)^2}{(H_S)^2} = 24.$$
If this is an equality, then $C \sim nH_S$, for some $n \in \mathbb{Z}$, and we have proved that this is impossible. So we have $C^2 < 24$, and since the self-intersection of a curve on a K3 surface is even, we get $C^2 \leq 22$. It follows that the arithmetic genus $p_a(C)$ of C satisfies $C^2 = 2p_a(C) - 2$, i.e., $p_a(C) \leq 12$. Thus, C cannot have more than 12 singular points. If C has 12 singular points, since all orbits of length 12 are in Q, we have $12 = Q \cdot C \geq 2 \cdot 12 = 24$, which is a contradiction.
6. We have proved that $p_a(C) \leq 12$ and that C is smooth, so its genus $g(C) \leq 12$. Note that C only contains one orbit of length 12 since $C \cdot Q = 12$. Using a classification of genera of smooth irreducible curves with $\mathfrak{A}_5$-actions and their orbit structures, for example in (Cheltsov & Shramov, 2016a, Lemma 5.1.5), we deduce that $g(C) \in \{0, 5, 10\}$.
7. Consider the action of $\mathfrak{S}_5$ given by the permutations of the coordinates leaving X and S invariant. By Cheltsov, Sarikyan, and Zhuang (2023), there is no $\mathfrak{S}_5$-invariant irreducible curve of degree 12 not contained in Q. Let C' be the other curve in the $\mathfrak{S}_5$-orbit of C. Since $C + C'$ is of degree 24 and since $\mathrm{Pic}^{\mathfrak{S}_5}(S) = \mathbb{Z} \cdot H_S$, we get $C + C' \sim 4H_S$.

□

4.3.2 Invariant curves not contained in Q

This subsection is devoted to proving the following.

Proposition 4.3.10. *If C is a G-invariant curve in X not contained in Q, then each irreducible component of C is not a non-canonical center of $(X, \lambda \mathscr{M}_X)$.*

Proof. This follows from Lemmas 4.3.11 and 4.3.12. □

We start with the case of irreducible curves. The method of the proof will be applied several times in this paper.

Lemma 4.3.11. *If C is an irreducible G-invariant curve not contained in Q, then C is not a non-canonical center of $(X, \lambda \mathscr{M}_X)$.*

Proof. By Lemma 4.3.9, the curve C is of degree 12, and there exists a unique smooth K3 surface S in the pencil $\mathscr{P}$ such that $C \subset S$, and the genus $g = g(C) \in \{0,5,10\}$. Let H be a general hyperplane section on X, and H_S its restriction to S. Assume that C is a non-canonical center of $(X, \lambda \mathscr{M}_X)$. Then $\mathrm{mult}_C(\lambda \mathscr{M}_X) > 1$. We have

$$\lambda \mathscr{M}_X|_S \sim_{\mathbb{Q}} mC + \Delta, \quad m \geq \mathrm{mult}_C(\lambda \mathscr{M}_X) > 1$$

for some divisor Δ on S not supported along C. In particular, the divisors

$$3H_S - C \sim_{\mathbb{Q}} \Delta + (m-1)C \quad \text{and} \quad 3H_S - mC \sim_{\mathbb{Q}} \Delta$$

are effective. By Lemma 4.3.9, there exists an irreducible curve C' such that C' is isomorphic to C and $C' \sim_{\mathbb{Q}} 4H_S - C$.

1. Assume that $g = 0$. We have $(C')^2 = (4H_S - C)^2 = -2$. So the divisor $4H_S - C$ is on an extremal ray of the Mori cone of S. Since H_S is ample, it implies that $C' - H_S \sim_{\mathbb{Q}} 3H_S - C$ is not rationally equivalent to any effective divisor. Hence, we get a contradiction.
2. Assume that $g = 5$. Notice that C' is nef since it is an irreducible curve on a smooth surface, and $(C')^2 = 2g(C') - 2 = 8$. But

$$(3H_S - C) \cdot C' = (3H_S - C) \cdot (4H_S - C) = -4 < 0,$$

which gives a contradiction.
3. Assume that $g = 10$. Let us first show that the linear system $|3H_S - C|$ has no fixed part. Notice that its mobile part is at least a pencil. Indeed, by Riemann-Roch theorem, we have

$$h^0(3H_S - C) \geq 2 + \frac{1}{2}(3H_S - C)^2 = 2.$$

So, if it has a base curve, it is of degree lower than 6. But there is no such G-invariant curve not contained in Q. The linear system $|3H_S - C|$ also does not have any fixed point. Indeed, we have $(3H_S - C)^2 = 0$, so the curves in this linear system are disjoint. Hence, there is no base curve in $|3H_S - C|$ other than C and it is nef. But $(3H_S - C) \cdot (3H - mC) < 0$, which yields a contradiction.

□

We exclude reducible curves in a similar way.

Lemma 4.3.12. *If C is a reducible G-invariant curve of degree 12 not contained in Q, then each irreducible component of C is not a non-canonical center of $(X, \lambda\mathscr{M}_X)$.*

Proof. By Lemma 4.3.5, C is the union of one of the two orbits $\mathscr{C}_6$ and $\mathscr{C}_6'$ of 6 conics. Note that $\mathscr{C}_6$ and $\mathscr{C}_6'$ are exchanged by the $\mathfrak{S}_5$-permutation action. Without loss of generality, assume that C is the union of conics in $\mathscr{C}_6$ and components of C are non-canonical centers of $(X, \lambda\mathscr{M}_X)$. Let C' be the union of conics in $\mathscr{C}_6'$. By Remark 4.3.8, $C \cup C'$ is contained in the smooth K3 surface $S = S_{2,25} \in \mathscr{P}$ under the notation of Lemma 4.3.7. Let H_S be a general hyperplane on S. Similarly as in Lemma 4.3.11, we know that

$$3H_S - mC$$

is an effective divisor for some $m > 1$. Using equations, we find that $C + C' \sim_{\mathbb{Q}} 4H_S$. Note that C' is on the border of the Mori cone of S, since it is the disjoint union of six conics where each of them has self-intersection -2. So $C' - H_S \sim_{\mathbb{Q}} 3H_S - C$ is not pseudo-effective, which contradicts the effectiveness of $3H_S - mC$. □

4.3.3 Points outside Q

Lemma 4.3.13. *Let $P \in X$ and Σ be its G-orbit. If $P \notin Q$ and $|\Sigma| \neq 5$, then P is not a center of non-canonical singularities of $(X, \lambda\mathscr{M}_X)$.*

Proof. Assume that P is a non-canonical center of $(X, \lambda\mathscr{M}_X)$. We consider two cases:

Case 1: When $|\Sigma| \geq 20$. Remark 1.6.11 implies that $(X, \frac{3}{2}\lambda\mathscr{M}_X)$ is not log-canonical at P. Let Λ be the non-log-canonical locus of $(X, \frac{3}{2}\lambda\mathscr{M}_X)$, and Λ_0 its zero-dimensional component.

Assume that a G-invariant curve C is contained in Λ. Consider two general elements $M_1, M_2 \in \mathscr{M}_X$, we have that

$$\frac{9}{4}\lambda^2(M_1 \cdot M_2) = mC + \Delta, \quad m \geq (\mathrm{mult}_C(\frac{3}{2}\lambda\mathscr{M}_X))^2$$

for an effective divisor Δ whose support does not contain C. Intersecting with a general hyperplane section H on X, we obtain that

$$\frac{81}{2} = H \cdot \frac{9}{4}\lambda^2(M_1 \cdot M_2) \geq m\deg(C).$$

By Theorem 1.6.8, we know that $m > 4$ and it follows that

$$\deg(C) \leq 10.$$

Lemma 4.3.6 implies that $C \subset Q$. By assumption, we have $\Sigma \not\subset Q$. Therefore, we know that $\Sigma \subset \Lambda_0$.

Let $\mathscr{I} = \mathscr{I}(X, \frac{3}{2}\lambda\mathscr{M}_X)$ be the multiplier ideal sheaf of $\frac{3}{2}\lambda\mathscr{M}_X$ on X. Note that

$$K_X + \frac{3}{2}\mathscr{M}_X + \frac{1}{2}\mathscr{O}_X(1) \sim_{\mathbb{Q}} \mathscr{O}_X(2).$$

Then, by Nadel vanishing theorem (cf. Theorem 1.6.12), we know that $h^1(X, \mathscr{I} \otimes \mathscr{O}_X(2)) = 0$ and it follows that

$$20 \leq |\Sigma| \leq |\mathrm{Supp}(\mathscr{I})| \leq h^0(\mathscr{O}_X(2)) = 14,$$

which is a contradiction.

Case 2: When $|\Sigma| < 20$, then by the classification of orbits we know that $|\Sigma| = 10$. This case is excluded by (Cheltsov, Sarikyan, & Zhuang, 2023, Proof of Proposition 3.4). The proof there applies verbatim. □

4.3.4 Points inside Q

Here we finish the proof of Proposition 4.3.1 by finding the G-equivariant α-invariant of Q.

Lemma 4.3.14. *One has $\alpha_G(Q) = \frac{3}{2}$.*

Proof. By Lemma 4.3.6, we see that the Bring curve B_6 of bidegree $(3,3)$ is the G-invariant divisor in Q with the least degree. By definition of the α-invariant, we have $\alpha_G(Q) \leq \frac{3}{2}$. Assume that $\alpha_G(Q) < \frac{3}{2}$. Then there exists a G-invariant effective $\mathbb{Q}$-divisor D on S such that

$$D \sim_Q \mathscr{O}_Q(3,3)$$

and (Q,D) is not log-canonical. Let Λ be the non-log-canonical locus of (Q,D). Assume that Λ contains a curve $C \subset Q$. We have $D = mC + \Delta$ where $m > 1$ and Δ is an effective divisor whose support does not contain C. Intersecting with a general hyperplane section H, we obtain

$$6 = H \cdot D \geq m \deg(C).$$

It follows that $\deg(C) < 6$. By Lemma 4.3.6, such curves do not exist.

Thus, Λ is 0-dimensional. We have $|\Lambda| \geq 12$ since orbits of length 5 and 10 are not in Q. Let $\varepsilon \in \mathbb{Q}^{>0}$ such that $(Q, (1-\varepsilon)D)$ is not klt at points in Λ, and $\mathscr{I}$ the multiplier ideal sheaf of $(1-\varepsilon)D$. Note that

$$K_Q + (1-\varepsilon)D + 3\varepsilon\mathscr{O}_Q(1,1) \sim_{\mathbb{Q}} \mathscr{O}_Q(1,1).$$

Applying Nadel vanishing theorem (cf. Theorem 1.6.12), we obtain

$$12 \leq |\mathrm{Supp}(\mathscr{I})| \leq h^0(\mathscr{O}_Q(1,1)) = 4,$$

which is absurd. So we obtain a contradiction and $\alpha_G(Q) = \frac{3}{2}$. □

Corollary 4.3.15. *Let Z be a non-canonical center of the pair $(X, \lambda\mathscr{M}_X)$, then $Z \not\subset Q$.*

Proof. If Z is contained in Q, then by inversion of adjunction, Z is a non-log-canonical center of $(Q, \lambda\mathscr{M}_X|_Q)$, which contradicts Lemma 4.3.14. □

4.4 The standard $\mathfrak{A}_5$-action on the cubic threefold

In this section, we study the cubic threefold $Y \subset \mathbb{P}^4_{x_1,\ldots,x_4}$ given by

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + \\ + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 = 0$$

with the same $G = \mathfrak{A}_5$-action through permutations of coordinates. Note that $\mathrm{Sing}(Y)$ consists of 5 nodes. The aim of this section is to prove the following result.

Proposition 4.4.1. *Let $\mathscr{M}_Y$ be a non-empty mobile G-invariant linear system on Y, and let $\mu \in \mathbb{Q}$ such that $\mu\mathscr{M}_Y \sim_{\mathbb{Q}} -K_Y$. Then the log pair $(Y, \mu\mathscr{M}_Y)$ is canonical away from $\mathrm{Sing}(Y)$.*

Proof. This follows from Propositions 4.4.6 and 4.4.7, and Corollary 4.4.9. □

Remark 4.4.2. If a curve C is a center of non-canonical singularities, then for any two general members $M_1, M_2 \in \mathscr{M}_Y$, we have that

$$\lambda^2(M_1 \cdot M_2) = mC + \Delta$$

for some $m > 1$ and some effective divisor Δ not supported along C. Intersecting with a general hyperplane H, we obtain that

$$12 = \lambda^2(M_1 \cdot M_2 \cdot H) > \deg(C). \tag{6}$$

Thus, we need to consider G-orbits of lengths less than 20 and G-invariant curves of degrees lower than 12. As in the previous section, we split into subsections according to whether or not a potential non-canonical center of $(Y, \mu\mathscr{M}_Y)$ belongs to the G-invariant hyperplane section.

4.4.1 Small G-orbits and G-invariant curves of low degrees

We begin with identifying small G-orbits and G-invariant curves of low degrees in Y.

Lemma 4.4.3. *A G-orbit of points in Y with length < 20 is one of the following:*

$$\begin{aligned}
\Sigma_5^1 &= \textit{the orbit of} \quad [1:0:0:0:0],\\
\Sigma_5^2 &= \textit{the orbit of} \quad [-2:3:3:3:3],\\
\Sigma_{10}^1 &= \textit{the orbit of} \quad [1:1:0:0:0],\\
\Sigma_{10}^2 &= \textit{the orbit of} \quad [1:-1:0:0:0],\\
\Sigma_{10}^3 &= \textit{the orbit of} \quad [-6-2\sqrt{6}:-6-2\sqrt{6}:6:6:6],\\
\Sigma_{10}^4 &= \textit{the orbit of} \quad [-6+2\sqrt{6}:-6+2\sqrt{6}:6:6:6],\\
\Sigma_{12}^1 &= \textit{the orbit of} \quad [1:\zeta_5:\zeta_5^2:\zeta_5^3:\zeta_5^4],\\
\Sigma_{12}^2 &= \textit{the orbit of} \quad [1:\zeta_5^2:\zeta_5^4:\zeta_5:\zeta_5^3],\\
\Sigma_{15} &= \textit{the orbit of} \quad [0:-1:-1:1:1],
\end{aligned}$$

where the length of each orbit is indicated by the subscript.

With the notation above, $\mathrm{Sing}(Y) = \Sigma_5^1$. There is a unique G-invariant hyperplane section in Y, given by

$$R := \{x_1 + x_2 + x_3 + x_4 + x_5 = 0\} \cap Y.$$

Note that R is the Clebsch cubic surface. One can check that

$$\Sigma_{10}^2, \Sigma_{12}^1, \Sigma_{12}^2, \Sigma_{15} \in R, \quad \Sigma_5^1, \Sigma_5^2, \Sigma_{10}^1, \Sigma_{10}^3, \Sigma_{10}^4 \notin R. \tag{7}$$

We recall some facts about the $\mathfrak{A}_5$-equivariant geometry of R, see (Cheltsov & Shramov, 2016b, Section 6.3) for more details. The surface R is G-linearizable. Indeed, there are two unions L_6, L_6' of 6 pairwise disjoint lines in R. Respective contractions of L_6 and L_6' give two G-birational maps $\pi, \pi' : R \to \mathbb{P}^2$. There is a unique G-invariant conic in $\mathbb{P}^2$. We denote its strict transforms under π and π' by C_6 and C_6' respectively.

Lemma 4.4.4. *Let C be a G-invariant curve in Y with $\deg(C) < 10$. Then $C \subset R$, $\deg(C) = 6$, and C is one of the following*

$$L_6, L_6', C_6, C_6', \quad \textit{or the Bring curve } B_6 \textit{ defined by (5).}$$

Proof. If $C \not\subset R$, then $C \cdot R = \deg(C) < 10$. By (7), we know that this is impossible. Thus $C \subset R$. The rest of the lemma follows from (Cheltsov & Shramov, 2016b, Theorem 6.3.18). □

Lemma 4.4.5. *Let C be a G-invariant curve in Y, of degree 10 and not contained in R. Then C is the union of 10 lines in the G-orbit of*

$$\{x_3 = x_4 = x_5 = 0\} \subset Y.$$

Moreover, these lines are the lines that pass through pairs of points in the singular locus of Y.

Proof. We may assume that C is G-irreducible. When C is an irreducible curve, by computation, we check that there is no G-invariant irreducible curve with a generic stabilizer. So G acts faithfully on C. Note that $C \cdot R = \deg(C) = 10$. By (7), we see that $C \cap R = \Sigma_{10}^2$ where all 10 points are smooth points of C. The stabilizer of a point in $C \cap R$ is $\mathfrak{S}_3$, which is a contradiction, since it should act faithfully in the tangent space of C at this point. It follows that C is a reducible curve.

So C can be 5 conics or 10 lines. Assume that C consists of 5 conics. Each conic spans a plane in $\mathbb{P}^4$, left invariant by $\mathfrak{A}_4 \subset G$. Each such plane intersects X along the conic and a residual line. Therefore, we obtain a G-orbit of 5 lines. One can check that there is no such orbit of lines in X. Similarly, we find that there is only one G-orbit of 10 lines, as is given in the assertion.

□

4.4.2 Invariant curves not contained in R

With the classification of G-irreducible invariant curves, we exclude curves not contained in R as non-canonical centers in this case.

Proposition 4.4.6. *Let C be a G-invariant curve in Y not contained in R. Then each irreducible component of C is not a non-canonical center of the pair $(Y, \mu\mathscr{M}_Y)$.*

Proof. Assume that the irreducible components of C are non-canonical centers. By Remark 4.4.2, we have $\deg(C) < 12$. From (7), we see that

$$\deg(C) = 10$$

and C is the union of 10 lines given in Lemma 4.4.5. This is impossible by (Cheltsov, Sarikyan, & Zhuang, 2023, Proof of Proposition 3.5). □

4.4.3 Points outside R

Proposition 4.4.7. *Let P be a point outside R, and Σ its G-orbit. If $\Sigma \neq \Sigma_5^1$, then P is not a non-canonical center of $(Y, \mu\mathscr{M}_Y)$.*

Proof. Assume that P is a non-canoncial center of $(Y, \mu\mathscr{M}_Y)$. By Remark 1.6.11, we know that P is a non-log-canonical center of $(Y, \frac{3}{2}\mu\mathscr{M}_Y)$. Let ε be a positive rational number such that

$$\Sigma \subset \Omega, \quad \Omega := \mathrm{Nklt}(Y, (\frac{3}{2} - \varepsilon)\mu\mathscr{M}_Y)$$

where Ω is the non-klt locus of $(Y, (\frac{3}{2} - \varepsilon)\mu\mathscr{M}_Y)$.

Assume that there is a curve $C \subset \Omega$. Let $M_1, M_2 \in (\frac{3}{2} - \varepsilon)\mu\mathscr{M}_Y$ and H a general hyperplane section of Y. Similarly as before, we have

$$27 \geq H \cdot (\frac{3}{2} - \varepsilon)^2 \mu^2 (M_1 \cdot M_2) \geq m \deg(C) > 4\deg(C)$$

for some number $m > 4$ by Theorem 1.6.8. Lemma 4.4.4 implies that $\deg(C) = 6$ and $C \subset R$. This shows that every curve in Ω is in R. It follows that the 0-dimensional component Ω_0 of Ω is non-empty since $P \notin R$. In particular, $\Omega_0 \supset \Sigma$. Observe that

$$K_Y + (\frac{3}{2} - \varepsilon)\mu\mathscr{M}_Y + 2\varepsilon\mathscr{O}_Y(1) \sim_{\mathbb{Q}} \mathscr{O}_Y(1).$$

Let $\mathscr{I}$ be the multiplier ideal sheaf of $(\frac{3}{2} - \varepsilon)\mu\mathscr{M}_Y$. By Nadel vanishing theorem (Theorem 1.6.12), we have $h^1(\mathscr{I} \otimes \mathscr{O}_Y(1)) = 0$. This implies that

$$|\Omega_0| \leq h^0(\mathscr{O}_Y(1)) = 5.$$

It follows that $\Omega_0 = \Sigma = \Sigma_5^1$ or Σ_5^2. The latter is impossible by (Cheltsov, Sarikyan, & Zhuang, 2023, Proposition 3.5). □

4.4.4 Points inside R

Similarly as in the previous section, it suffices to find the G-equivariant α-invariant of R.

Lemma 4.4.8. *One has $\alpha_G(R) = 2$.*

Proof. Note that $B_6 \subset R$ is a G-invariant effective divisor such that $B_6 \sim_{\mathbb{Q}} -2K_R$. It follows that $\alpha_G(R) \leq 2$. Suppose that $\alpha_G(R) < 2$. Then there exists a G-invariant effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -2K_R$ such that (R, D) is not log-canonical. Let Λ be the non-log-canonical locus of (R, D). Let $\varepsilon \in \mathbb{Q}_{>0}$ such that the non-klt locus Ω of $(R, (1-\varepsilon)D)$ contains Λ. Assume that Ω contains some curve C', then

$$(1-\varepsilon)D = mC' + \Delta, \quad m \geq 1$$

for some effective 1-cycle Δ whose support does not contain C'. Intersecting with a general hyperplane section H on R, we obtain

$$6 > H \cdot (1-\varepsilon)D = H \cdot (mC' + \Delta) \geq \deg(C'),$$

which is impossible by Lemma 4.4.4.

Thus, Ω consists of finitely many points. Let $n = |\Lambda|$ and $\mathscr{I}$ be the multiplier ideal sheaf of $(1-\varepsilon)\mu\mathscr{M}_R$. Observe that

$$K_R + (1-\varepsilon)D + 2\varepsilon\mathscr{O}_R(1) \sim_{\mathbb{Q}} \mathscr{O}_R(1).$$

By Nadel vanishing theorem, we know that $h^1(\mathscr{O}_R(1)\otimes\mathscr{I}) = 0$ and

$$n = |\Lambda| \leq |\Omega| \leq h^0(\mathscr{O}_R(1)) = 4,$$

which implies that $n = 0$ since there is no G-orbit of length ≤ 4 in R. □

Corollary 4.4.9. *Let Z be a non-canonical center of the pair $(Y, \mu\mathscr{M}_Y)$, then $Z \not\subset R$.*

Proof. Assume that Z is contained in R. By inversion of adjunction, the pair $(R, \mu\mathscr{M}_X|_R)$ is not log-canonical, which contradicts Lemma 4.4.8. □

4.5 The nonstandard $\mathfrak{A}_5$-action on the quadric threefold

In this section, we study the non-standard $\mathfrak{A}_5$-action. Let $G = \mathfrak{A}_5$ acting on the smooth quadric threefold given by

$$X = \left\{ \sum_{1\leq i\leq j\leq 5} x_i x_j = 0 \right\} \subset \mathbb{P}^4 \tag{8}$$

with the G-action generated by

$$\begin{aligned}(\mathbf{x}) &\mapsto (x_4, x_1, x_5, x_2, -x_1-x_2-x_3-x_4-x_5),\\ (\mathbf{x}) &\mapsto (x_4, -x_1-x_2-x_3-x_4-x_5, x_1, x_3, x_2).\end{aligned} \tag{9}$$

The aim of this section is to prove the following result.

Proposition 4.5.1. *Let $\mathscr{M}_X$ be a non-empty mobile G-invariant linear system on X, and $\lambda \in \mathbb{Q}$ such that $\lambda\mathscr{M}_X \sim_{\mathbb{Q}} -K_X$. Let Z be a G-irreducible subvariety whose components are centers of non-canonical singularities of $(X, \lambda\mathscr{M}_X)$. Then Z is one of the following:*

- *the union of 5 points in the orbit Σ_5 or Σ_5' given in Lemma 4.5.3,*
- *the rational curve C_4 or C_4' of degree 4 given by* (12)*,*
- *the rational curve C_8 or C_8' of degree 8 described in Remark 4.5.16,*
- *(possibly) a smooth irreducible curve of degree 10 and genus 6.*

Proof. This follows from Proposition 4.5.6 and Proposition 4.5.22. □

Remark 4.5.2. Equations of C_4, C_4', C_8, C_8' can be found in Pinardin and Zhang (2025a). Sarkisov links centered at these curves are presented in later subsections. We do not know the existence of the curve of degree 10. This is not necessary for our main result, see Lemma 4.5.13.

Note that the G-action on the ambient $\mathbb{P}^4$ arises from the unique 5-dimensional irreducible linear representation of G. The nature of this action creates more challenges for our classifications since there are more possibilities of G-invariant curves and non-canonical centers, cf. Proposition 4.3.1. The $\mathfrak{A}_5$-equivariant geometry of K3 surfaces turns out to be crucial to our analysis in this section.

First, we classify G-orbits of lengths less than 20, and the G-irreducible invariant curves of degrees at most 17. In the second subsection, we will study the singularities of pairs $(X, \lambda\mathscr{M}_X)$ along G-invariant curves, and in the third subsection, we will study them along G-orbits.

4.5.1 Small G-orbits and G-invariant curves of low degrees

Lemma 4.5.3. *A G-orbit of points in X with length < 20 is one of the following:*

$$\begin{aligned} \Sigma_5 &= \text{the orbit of} \quad [1 : \zeta_6 - 1 : -\zeta_6 : \zeta_6 - 1 : 1], \\ \Sigma_5' &= \text{the orbit of} \quad [1 : -\zeta_6 : \zeta_6 - 1 : -\zeta_6 : 1], \\ \Sigma_{12} &= \text{the orbit of} \quad [\zeta_5^3 : \zeta_5^2 : 0 : \zeta_5 : 1], \\ \Sigma_{12}' &= \text{the orbit of} \quad [\zeta_5^4 : \zeta_5 : 0 : \zeta_5^3 : 1], \end{aligned}$$

where the length of each orbit is indicated by the subscript.

Lemma 4.5.4. *Let C be a G-invariant curve in X with $\deg(C) \le 17$. Then C has trivial generic stabilizer, i.e, the G-orbit of a general point in C has length 60.*

Proof. By direct computation, one sees that the only irreducible curves in X with non-trivial generic stabilizers are lines and conics, and their G-orbits have lengths 20 and 15 respectively. □

Lemma 4.5.5. *Let C be a G-invariant reducible curve of degree at most 17. Then C is the union of curves in one of the following orbits:*

- *an orbit of 5 conics*

$$\mathscr{C}_5 = \text{orbit of } \{x_1 + x_4 = x_2 + x_3 = 0\} \cap X,$$

- *one of the following 2 orbits of 6 conics*

$$\mathscr{C}_6 = \text{orbit of } C_1, \qquad \mathscr{C}_6' = \text{orbit of } C_2,$$

where

$$a = \zeta_5 + \zeta_5^4,$$

$$C_1 = \{x_1 - x_3 - a(x_4 - x_5) = x_2 + a(x_3 - x_4) - x_5 = 0\} \cap X,$$

$$C_2 = \{x_1 - x_3 + (1+a)(x_4 - x_5) = x_2 - (1+a)(x_3 - x_4) - x_5 = 0\} \cap X,$$

- *one of the following 2 orbits of 12 lines*

$$\mathscr{L}_{12} = \textit{the orbit of} \quad \{\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} \zeta_5^{i-1} x_i = \sum_{i=1}^{5} \zeta_5^{3(i-1)} x_i = 0\},$$

$$\mathscr{L}'_{12} = \textit{the orbit of} \quad \{\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} \zeta_5^{i-1} x_i x_i = \sum_{i=1}^{5} \zeta_5^{2(i-1)} x_i = 0\}.$$

Each of the orbits above consists of pairwise disjoint curves.

Proof. The proof is similar to that of Lemma 4.3.5. □

4.5.2 Invariant curves

Similarly as in Section 4.3 (see Remark 4.3.2), if a curve C is in the non-canonical center of $(X, \lambda\mathscr{M}_X)$, we have

$$\deg(C) \leq 17.$$

The main result of this subsection is:

Proposition 4.5.6. *Let $\mathscr{M}_X$ be a non-empty mobile G-invariant linear system on X, and $\lambda \in \mathbb{Q}$ such that $\lambda\mathscr{M}_X \sim_{\mathbb{Q}} -K_X$. If a curve C is a non-canonical center of $(X, \lambda\mathscr{M}_X)$, then C is an irreducible curve of degree 4,8, or 10, and is one of the curves described in Proposition 4.5.1.*

Proof. We explain how the results in this subsection show the assertion. Lemma 4.5.17 shows that if C is contained in certain surfaces R or R' explicitly given by (11), then $\deg(C) = 4$. When C is not in R or R', Lemma 4.5.9 shows $\deg(C) \in \{8, 10, 12, 16\}$. Lemma 4.5.11 excludes the case $\deg(C) = 12$. Lemmas 4.5.12 and 4.5.19 show that $\deg(C) = 16$ is also impossible. Then Lemmas 4.5.13, 4.5.15, 4.5.17 and 4.5.18 prove that all such curves are among those described in Proposition 4.5.1. □

First, we present curves of degree 4. Consider the pencil consisting of G-invariant cubics in $\mathbb{P}^4$ given by

$$\{a_1 f_1 + a_2 f_2 = 0\} \subset \mathbb{P}^4_{x_1,\ldots,x_5}, \quad [a_1 : a_2] \in \mathbb{P}^1$$

where

$$f_1 = x_1^2 x_2 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_2^2 x_3 + x_2 x_3^2 + 2x_2 x_3 x_4 + x_3^2 x_4 + x_3 x_4^2 + \\ + x_1^2 x_5 + 2x_1 x_2 x_5 + 2x_1 x_4 x_5 + 2x_3 x_4 x_5 + x_4^2 x_5 + x_1 x_5^2 + x_4 x_5^2, \quad (10)$$

$$f_2 = x_1^2x_3 + x_1x_3^2 + x_1^2x_4 + 2x_1x_2x_4 + x_2^2x_4 + 2x_1x_3x_4 + x_1x_4^2 + x_2x_4^2 + \\ + x_2^2x_5 + 2x_1x_3x_5 + 2x_2x_3x_5 + x_3^2x_5 + 2x_2x_4x_5 + x_2x_5^2 + x_3x_5^2.$$

In particular, there are two G-invariant chordal cubics in $\mathbb{P}^4$, i.e., the cubic threefold whose singular locus is a twisted quartic curve. Their intersections with X are two non-normal surfaces, given by

$$R = \{(-\zeta_5^3 - \zeta_5^2 + 1)f_1 + f_2 = 0\} \cap X, \\ R' = \{(\zeta_5^3 + \zeta_5^2 + 2)f_1 + f_2 = 0\} \cap X. \tag{11}$$

Their intersection $R \cap R'$ is an irreducible curve whose singular locus is $\Sigma_{12} \cup \Sigma'_{12}$. Let

$$C_4 = \mathrm{Sing}(R), \quad C'_4 = \mathrm{Sing}(R'). \tag{12}$$

Then C_4 and C'_4 are quartic rational normal curves such that

$$\Sigma_{12} \in C_4 \setminus C'_4, \quad \Sigma'_{12} \in C'_4 \setminus C_4.$$

These two curves can be non-canonical centers of $(X, \lambda\mathscr{M}_X)$. Sarkisov links centered at them are involutions on X, presented in Lemma 4.5.15. Now, let $\mathscr{P}$ be the pencil consisting of G-invariant K3 surfaces S_{a_1,a_2} in X given by

$$S_{a_1,a_2} := \{a_1f_1 + a_2f_2 = 0\} \cap X, \qquad [a_1 : a_2] \in \mathbb{P}^1. \tag{13}$$

Remark 4.5.7. Each of the orbits in Lemma 4.5.5 is contained in a unique member in $\mathscr{P}$. We record

orbit	a_1	a_2
$\mathscr{C}_5$	1	1
$\mathscr{C}_6$	$63(\zeta_5^3 + \zeta_5^2) + 145$	89
$\mathscr{C}'_6$	$63(\zeta_5^3 + \zeta_5^2) + 145$	89
$\mathscr{L}_{12}$	$-\zeta_5^3 - \zeta_5^2 + 1$	1
$\mathscr{L}'_{12}$	$\zeta_5^3 + \zeta_5^2 + 2$	1

where each orbit in the first column is contained in $S_{a_1,a_2} \in \mathscr{P}$ for a_1, a_2 indicated in the same row.

Lemma 4.5.8. *A surface $S_{a_1,a_2} \in \mathscr{P}$ is singular if and only if*

1. $[a_1 : a_2] = [-3\zeta_6 + 8 : 7], \quad \mathrm{Sing}(S_{a_1,a_2}) = \Sigma_5,$
2. $[a_1 : a_2] = [3\zeta_6 + 5 : 7], \quad \mathrm{Sing}(S_{a_1,a_2}) = \Sigma'_5,$
3. $[a_1 : a_2] = [-\zeta_5^3 - \zeta_5^2 + 1 : 1], \quad \mathrm{Sing}(S_{a_1,a_2}) = C_4,\ S_{a_1,a_2} = R,$
4. $[a_1 : a_2] = [\zeta_5^3 + \zeta_5^2 + 2 : 1], \quad \mathrm{Sing}(S_{a_1,a_2}) = C'_4,\ S_{a_1,a_2} = R'.$

Moreover, when $S_{a_1,a_2} \in \mathscr{P}$ is smooth, it contains no orbit of length 5.

Proof. First, we consider the case when $S = S_{a_1,a_2}$ is normal. The singular locus $\mathrm{Sing}(S_{a_1,a_2})$ forms a G-invariant set. Note that $K_S \sim 0$. If S has non-du Val singularities, it has at least 12 of them, since the smallest orbit on X has length 12. This is impossible. Thus, S has at worst du Val singularities and its minimal resolution is a smooth K3 surface, whose Picard rank is bounded by 20. Then $|\mathrm{Sing}(S_{a_1,a_2})| < 20$ and $\mathrm{Sing}(S_{a_1,a_2})$ consists of orbits in Lemma 4.5.3. One can check that the four cases in the assertion are the only possible cases. Note that being singular along Σ_{12} or Σ'_{12} forces S_{a_1,a_2} to be non-normal.

Now assume that $S = S_{a_1,a_2}$ is singular along a curve Z. Let S' be a general member of $\mathscr{P}$. Recall that $S' \cap S$ is an irreducible curve whose singular locus is $\Sigma_{12} \cup \Sigma'_{12}$. It follows that $\emptyset \neq Z \cap S \subset \Sigma_{12} \cup \Sigma'_{12}$ and the only possible cases are $S = R$ or R'. □

Lemma 4.5.9. *Let C be a G-invariant curve not contained in $R \cup R'$ such that $\deg(C) \leq 17$. Then the following assertions hold.*

1. *We have $\deg(C) \in \{8, 10, 12, 16\}$.*
2. *The irreducible components of C are pairwise disjoint.*
3. *In the pencil $\mathscr{P}$, there exists a unique surface S containing C.*
4. *If C is irreducible, then C is a Cartier divisor on S.*
5. *The surface S is smooth.*
6. *If $\deg(C) \neq 16$, then C is smooth.*
7. *If $\deg(C) \neq 16$ and C is irreducible, then its genus satisfies*

$$g(C) = \begin{cases} 0 & \text{if } \deg(C) = 8 \text{ or } 12, \\ 6 & \text{if } \deg(C) = 10. \end{cases}$$

Proof. We may assume that C is G-irreducible.

1. Since C is not contained in R, we have $C \cdot R = 3\deg(C) \leq 51$, and $C \cap R$ splits into G-orbits. Hence,

   $$C \cdot R = 12a + 20b + 30c = 3\deg(C) \leq 51, \quad a, b, c \in \mathbb{Z}_{\geq 0}.$$

   If $a > 0$, since R is singular at Σ_{12}, we have $2 \leq a \leq 4$ and $b = c = 0$. Then $\deg(C) \in \{8, 12, 16\}$. If $a = 0$, then $b = 0$ and $c = 1$. In this case, $\deg(C) = 10$.
2. This is obvious if C is irreducible. When C is reducible, the assertion follows from the classification in Lemma 4.5.5.
3. Let P be a general point of C. There exists a unique surface S in $\mathscr{P}$ that passes through P. The intersection $S \cap C$ contains the orbit of P, which has length 60 by Lemma 4.5.4. If C is not contained in S, we have that $C \cdot S = 3\deg(C) \in \{24, 30, 36, 48\}$, which is a contradiction. Therefore $C \subset S$.

4. Assume that $C \cap \mathrm{Sing}(S) \neq \emptyset$, otherwise C is Cartier. Let $f\colon \widetilde{S} \to S$ be the blowup of $C \cap \mathrm{Sing}(S)$. Denoting by $\widetilde{C}$ the strict transform of the curve C by f, we have $\widetilde{C} \sim_{\mathbb{Q}} f^*(C) - mE$ for some $m \in \frac{1}{2}\mathbb{Z}$, where E is the exceptional divisor of f. Consider a point $P \in C \cap \mathrm{Sing}(S)$, and a component E_P of E mapped to P. We have $\widetilde{C} \cdot E_P = 2m$. By assumption, we have $S \neq R$ and R'. Lemma 4.5.8 implies that $|C \cap \mathrm{Sing}(S)| = 5$ and the stabilizer of P is $\mathfrak{A}_4$, which acts faithfully on $E_P = \mathbb{P}^1$. It follows that $2m = 4a + 6b + 12c$, for some non-negative integers a, b, c, because the possible lengths of $\mathfrak{A}_4$-orbits on $\mathbb{P}^1$ are 4,6, and 12. Hence m is an integer, and C is Cartier.
5. When C is reducible, this follows from Remark 4.5.7. We assume that C is irreducible. The proof is similar to that of Lemma 4.3.9. Assume that S is singular, then $\mathrm{rk}(\mathrm{Pic}^G(S)) = 1$. Let H be a general hyperplane section on X, and H_S its restriction to S. Since $\deg(H_S) = 6$ is not a square, we know that $\mathrm{Pic}^G(S)$ is generated by H_S. Note that $C \in \mathrm{Pic}^G(S)$ because C is Cartier. It follows that $C \sim nH_S$ for some integer n. We know that $\deg(C) = C \cdot H_S = 6n$, which implies that $\deg(C) = 12$ and $n = 2$. Recall that the only G-invariant quadric hypersurface in $\mathbb{P}^4$ is X. So there is no G-invariant curve linear equivalent to $2H_S$ and we obtain a contradiction.
6. When C is reducible, this follows from Lemma 4.5.5. We assume that C is irreducible and singular. Let $\widetilde{C}$ be a minimal resolution of singularities of C. Since S is smooth, then by Lemma 4.5.8, we know that S does not contain an orbit of length 5. We get

$$g(C) = g(\widetilde{C}) = p_a(\widetilde{C}) = p_a(C) - 12a - 20b - 30c - 60d,$$

where a, b, c, d are non-negative integers which are not all 0, $g(C)$ is the geometric genus, and $p_a(C)$ is the arithmetic genus of C. Hodge index theorem gives

$$C^2 \leq \frac{(C \cdot H_S)^2}{(H_S)^2}. \tag{14}$$

If this is an equality, then $C \sim nH_S$, for some $n \in \mathbb{Z}$, and we have proved above that this is impossible. So (14) is a strict inequality, i.e.,

$$C^2 \leq \begin{cases} 10 & \text{if } \deg(C) = 8, \\ 16 & \text{if } \deg(C) = 10, \\ 22 & \text{if } \deg(C) = 12. \end{cases}$$

Recall that $p_a(C) = \frac{C^2+2}{2}$. We obtain that

$$p_a(C) \leq \begin{cases} 6 & \text{if } \deg(C) = 8, \\ 9 & \text{if } \deg(C) = 10, \\ 12 & \text{if } \deg(C) = 12. \end{cases} \tag{15}$$

Then, the only possibility is $a = 1, b = c = 0$ and

$$\deg(C) = 12, \quad p_a(C) = 12, \quad g(C) = 0.$$

In this case, we have $C \cdot B = 36$, where $B = R \cap R'$ is an irreducible curve of degree 18. On the other hand, since both C and B are singular at a common orbit of length 12, we have $C \cdot B \geq 12 \cdot 4 = 48$. We obtain a contradiction. So C is smooth.

7. We know that C is smooth, and we have the bound (15) on its genus. First, when $\deg(C) = 8$, we find all such curves in Lemma 4.5.18 and it follows that $g(C) = 0$. When $\deg(C) = 10$, recall that C contains no orbit of length 12. Then by (Cheltsov & Shramov, 2016b, Lemma 5.1.5), or by searching through the database of curves with $\mathfrak{A}_5$-actions in LMFDB Collaboration (2025), we find that $g(C) = 6$. Finally, if $\deg(C) = 12$, similarly going through the classification, we get $g(C) = 0$.

□

First, we exclude several curves as possible non-canonical centers.

Lemma 4.5.10. *Let C be a G-invariant union of r conics not contained in R or R', with $r \in \{5, 6\}$. Then each irreducible component of C is not a center of non-canonical singularities of $(X, \lambda \mathscr{M}_X)$.*

Proof. By Lemma 4.5.9, there exists a unique smooth K3 surface S in the pencil $\mathscr{P}$ containing C. Let H_S be a general hyperplane section of S and $m = \mathrm{mult}_C(\lambda \mathscr{M}_X)$. Assume that irreducible components of C are non-canonical centers. Then $m > 1$, and we have

$$\lambda \mathscr{M}_X|_S \sim_{\mathbb{Q}} mC + \Delta, \quad m \geq \mathrm{mult}_C(\lambda \mathscr{M}_X) > 1$$

for some effective divisor Δ on S not supported along C. It follows that the divisor

$$3H_S - C \sim_{\mathbb{Q}} \Delta + (m-1)C$$

is effective. On the other hand, consider the divisor on S given by

$$D = (r-1)H_S - C.$$

The equations of C are given in Lemma 4.5.5. By computation, we check that the linear subsystem in $|\mathscr{O}_X(r-1)|$ consisting of surfaces passing through C does not contain any base curve other than C. It follows that D is nef. However, we compute

$$D \cdot (3H_S - C) = \begin{cases} -8 & \text{if } r = 5, \\ -18 & \text{if } r = 6, \end{cases}$$

which gives a contradiction. □

Lemma 4.5.11. *Let C be a G-invariant curve of degree 12 not contained in R or R'. Then each irreducible component of C is not a center of non-canonical singularities of $(X, \lambda\mathscr{M}_X)$.*

Proof. If C is reducible, it is a union of 6 conics by Lemma 4.5.5. The assertion follows from Lemma 4.5.10. We assume that C is irreducible.

By Lemma 4.5.9, the curve C is rational, and there exists a smooth K3 surface S in the pencil $\mathscr{P}$ containing C. Let H_S be a general hyperplane section on S. We have $(4H_S - C)^2 = -2$, and Riemann-Roch theorem gives $h^0(4H_S - C) \geq 1$. Assume that it is an equality. Then the only element of $|4H_S - C|$ is the class of a G-invariant curve C' of degree 12. If C' is reducible, by Lemma 4.5.5, C' is either a union of 12 lines or 6 conics. None of these is possible: the 12 lines are contained in R or R'; and $C'^2 = -12$ if C' is a union of 6 conics. It follows that C' is irreducible, in which case we can exclude C' as a non-canonical center the same way as in the proof of Lemma 4.3.11.

Assume now that $h^0(4H_S - C) > 1$. We show that this is impossible. The linear system $|4H_S - C|$ splits into a fixed part $\mathscr{F}$ and a mobile part $\mathscr{G}$. But $(4H_S - C)^2 = -2$, so $|4H_S - C|$ is not nef. We deduce that $\mathscr{F}$ is not empty. Let $F \in \mathscr{F}$. The degree of F is 8 or 10. The latter case would imply that curves in $\mathscr{G}$ have degree 2, i.e., there is a pencial of rational curves in S, which is impossible on K3 surfaces. Assume that $\deg(F) = 8$. Then the degree of a general member $M \in \mathscr{G}$ is four. Either M is a smooth elliptic curve or M is rational. Again, the latter is impossible on K3 surfaces. So M is a smooth elliptic curve. Consider the matrix

$$A = \begin{pmatrix} H_S^2 & H_S \cdot F & H_S \cdot M \\ F \cdot H_S & F^2 & F \cdot M \\ M \cdot H_S & M \cdot F & M^2 \end{pmatrix}.$$

Since $\mathrm{Pic}^G(S)$ is of rank at most two by (Cheltsov & Shramov, 2016a, Proposition 6.7.3), the determinant of A must be zero. But we get

$$\det(A) = \det \begin{pmatrix} 6 & 8 & 4 \\ 8 & -2 & 7 \\ 4 & 7 & 0 \end{pmatrix} = 186,$$

hence we obtain a contradiction. □

We turn to G-invariant curves of degree 16, which are necessarily irreducible. The strategy of the proof is similar, but such curves may be singular. We first treat smooth curves here. The case of singular curves will be excluded at the end of this subsection, where we explicitly find all such curves in equations.

Lemma 4.5.12. *Let C be a smooth irreducible G-invariant curve of degree 16 in X that is not contained in $R \cup R'$. Then C is not a non-canonical center of $(X, \lambda \mathscr{M}_X)$.*

Proof. Let S be the unique smooth K3 surface in $\mathscr{P}$ containing C, and H_S a general hyperplane section on S. Similarly as in Lemma 4.5.11, we know that $3H_S - C$ is effective and we seek for a contradiction by finding a nef divisor on S intersecting $3H_S - C$ negatively.

The Hodge index theorem implies that

$$C^2 \leq \frac{(C \cdot H_S)^2}{(H_S)^2} = \frac{128}{3} \quad \Longrightarrow \quad C^2 \leq 42, \quad g \leq 22,$$

where $g = g(C)$ is the genus of C. Possible genera of smooth irreducible curves with $\mathfrak{A}_5$-actions and their orbit structures are classified in (Cheltsov & Shramov, 2016b, Lemma 5.1.5). Recall from the proof of Lemma 4.5.5 that C contains at least one G-orbit of length 12. By (Cheltsov & Shramov, 2016b, Lemma 5.1.5), we find that

$$g \in \{0, 4, 5, 9, 10, 13, 15, 19, 20\},$$

so $C^2 = 2g - 2 \in \{-2, 6, 8, 16, 18, 24, 28, 36, 38\}$. Put

$$n = \begin{cases} 4 & \text{if } g \in \{19, 20\}, \\ 5 & \text{if } g \in \{9, 10, 13, 15\}, \\ 6 & \text{if } g \in \{0, 4, 5\}. \end{cases}$$

One can check that $(nH_S - C)^2 \geq 0$, and

$$(nH_S - C) \cdot (3H_S - C) = 2n - 48 + C^2 < 0.$$

Therefore, if $nH_S - C$ is nef, we are done. Now let us show that $nH_S - C$ is nef for all possible genera.

Assume that $nH_S - C$ is not nef. By Riemann-Roch, $(nH_S - C)^2 \geq 0$ implies that $h^0(nH_S - C) \geq 2$. So $|nH_S - C|$ has a mobile part. Moreover, since $nH_S - C$ is not nef, $|nH_S - C|$ has a G-irreducible fixed component F such that $\deg(F) \leq \deg(nH_S - C)$. The curves in the mobile part cannot be rational because S is a K3 surface, and thus their degree is at least 4. It follows that

$$\deg(F) \leq \deg(nH_S - C) - 4 = \begin{cases} 4 & \text{if } g \in \{19, 20\}, \\ 10 & \text{if } g \in \{9, 10, 13, 15\}, \\ 16 & \text{if } g \in \{0, 4, 5\}. \end{cases}$$

Lemma 4.5.9 also implies that $\deg(F) \in \{8,10,12,16\}$. This immediately excludes the possibility $g \in \{19,20\}$. In the other two cases, since $\mathrm{rk}\,\mathrm{Pic}^G(S) \leqslant 2$, the intersection matrix of F, H_S and C is degenerate. In particular, let $x = F \cdot C$, we have

$$\det \begin{pmatrix} F^2 & x & \deg(F) \\ x & C^2 & 16 \\ \deg(F) & 16 & 6 \end{pmatrix} = 0, \tag{16}$$

which gives a quadratic equation in x. We show that this equation does not have integer solutions satisfying the geometric conditions. Since F is a fixed component, we know that $h^0(F) = 1$, and by Riemann-Roch, $F^2 < 0$. Hence, if F is irreducible, then, by adjunction formula, we have $F^2 = -2$. When F is reducible, F^2 is supplied by Lemma 4.5.5. In particular, we have

$$F^2 = \begin{cases} -12 & \text{if } \deg(F) = 12 \text{ and } F \text{ is reducible,} \\ -10 & \text{if } \deg(F) = 10 \text{ and } F \text{ is reducible,} \\ -2 & \text{if } F \text{ is irreducible,} \end{cases}$$

Now, running through all possibilities, we find that (16) has an integer solution only in the following two cases:

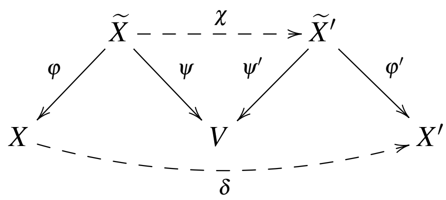
1. $n = 6$, $\deg(F) = 16$, $F^2 = -2$, $C^2 = -2$, $C \cdot F = -2$,
2. $n = 5$, $\deg(F) = 10$, $F^2 = -10$, $C^2 = 16$, $C \cdot F = 0$.

So, in the first case, we have $C = F$, and in the second case, C is a union of 5 conics. In both cases, we know that the linear system $|nH - C - F|$ is not empty since it has the same mobile part as $|nH - C|$. One can compute that $(nH - C - F)^2 < 0$, implying that $|nH - C - F|$ has a fixed component of degree 4, which is impossible. We obtain a contradiction, and this completes the proof. □

Now we turn to irreducible curves of degree 8 or 10. Such curves can indeed be non-canonical centers. We characterize the Sarkisov links arising from them. The following result will allow us to prove in Section 4.7 that up to some G-birational self-map of X which normalizes the image of G in $\mathrm{Aut}(X)$, the pair $(X, \lambda\mathscr{M}_X)$ is canonical away from $\Sigma_5 \cup \Sigma_5'$.

Lemma 4.5.13. *Let Z be a G-invariant smooth irreducible curve Z not contained in $R \cup R'$, of degree 8 and genus 0, or of degree 10 and genus 6. Assume that Z is a non-canonical center of $(X, \lambda\mathscr{M}_X)$. Let $\varphi : \widetilde{X} \to X$ be the blowup of Z. Then $-K_{\widetilde{X}}$ is big and nef. Moreover, for $n \gg 0$, the linear system $|n(-K_{\widetilde{X}})|$ is base point free and gives a small birational map $\psi : \widetilde{X} \to V$.*

There exists the following G-equivariant commutative diagram



(17)

where

1. χ *is a composition of flops,*
2. ψ' *is also a small birational map,*
3. φ' *is a $K_{\widetilde{X}'}$-negative extremal contraction,*
4. X' *is also a smooth quadric threefold, and φ' is the blowup of a curve $Z' \subset X'$ of the same degree and genus as Z,*
5. X *and X' are G-isomorphic, i.e. the birational map δ normalizes the image of G in* $\mathrm{Aut}(X)$.

Proof. First, we introduce some notation. Let $g(Z)$ be the genus of Z, E the exceptional divisor of φ, H a general hyperplane section on X, and $\widetilde{H}$ the pullback of H to $\widetilde{X}$. Let S be the unique K3 surface in $\mathscr{P}$ containing Z, $\widetilde{S}$ its strict transform on $\widetilde{X}$, and H_S the restriction of H to S. Note that $\widetilde{S} \simeq S$ since S is smooth.

To show that $-K_{\widetilde{X}}$ is big, we compute

$$(-K_{\widetilde{X}})^3 = (3\widetilde{H} - E)^3 = 2g(Z) - 6 \cdot \deg(C) + 52 = 4 > 0.$$

To show that $-K_{\widetilde{X}}$ is nef, it suffices to show that $|3H_S - Z|$ contains no other base curve than Z, i.e., $3H_S - Z$ is nef. Indeed, if $-K_{\widetilde{X}}$ is not nef, then the divisor $-K_{\widetilde{X}}|_{\widetilde{S}} = (3\widetilde{H} - E)|_{\widetilde{S}} = 3H_S - Z$ is also not nef.

Let us first do this when Z is a rational curve of degree 8. By Riemann-Roch, $h^0(3H_S - Z) = 2 + \frac{(3H_S - Z)^2}{2} = 4$, so the degree 10 linear system $|3H_S - Z|$ has a non-trivial mobile part. Hence a fixed curve F of this linear system has degree at most 9. Lemma 4.5.9 implies that F is of degree 8. But then the mobile part of $3H_S - Z$ contains a pencil of rational curves, which is impossible on K3 surfaces. We conclude that $(3H_S - Z)$ does not contain a base curve other than Z.

Similarly, if Z has degree 10 and genus 6. Riemann-Roch theorem implies that $h^0(3H_S - Z) = 2 + \frac{(3H_S - Z)^2}{2} = 4$, so the degree 8 linear system $|3H_S - Z|$ has a nontrivial mobile part. Hence, a fixed curve F of this linear system is of degree at most 7. Lemma 4.5.9 implies that this is impossible.

We prove that $-K_{\widetilde{X}}$ is big and nef. Then it follows from base point free theorem that the linear system $|n(-K_{\widetilde{X}})|$ is base point free for $n \gg 0$, and it gives a birational map $\psi : \widetilde{X} \to V$. Either ψ contracts a divisor, or ψ is small. However, the former case is impossible, because we assume that Z is a center of non-canonical singularities of $(X, \lambda \mathscr{M}_X)$. If ψ contracts a divisor, then this divisor must be a fixed component of the linear system $\mathscr{M}_X$, which is impossible, since $\mathscr{M}_X$ is mobile by assumption.

Hence, we see that ψ is a small birational contraction. Then the existence of G-equivariant commutative diagram and (1) – (3) follow from the Sarkisov program. This is a type II Sarkisov link. Moreover, (4) follows from matching the numerical invariants with a classification of such links in Cutrone and Marshburn (2025); Jahnke, Peternell, and Radloff (2005, 2011). Our cases correspond to rows 71 and 72 of Table 1 in Cutrone and Marshburn (2025). In particular, we find X' is also a smooth quadric. To show (5), we notice that any other action of $\mathfrak{A}_5$ on the quadric X' has an invariant hyperplane section, which would intersect Z' in $\deg(Z') = 8$ or 10 points. But, by Cheltsov and Shramov (2016a), the smallest possible orbit of $\mathfrak{A}_5$ on a smooth curve is of length 12, so we get a contradiction. □

Remark 4.5.14. Lemma 4.5.13 shows that if such a Sarkisov link exists, it necessarily leads to a G-isomorphic quadric and gives no new G-Mori fibre space. Remark 4.5.16 and Lemma 4.5.18 finds all curves of degree 8 satisfying the assumptions of Lemma 4.5.13. On the other hand, we do not know the existence of such a curve of degree 10.

Finally, we show that the only curves in R or R' which can be non-canonical centers are C_4 and C_4'. The Sarkisov links centered at them have been studied in (Araujo et al., 2023a, Section 5.9) and Malbon (2025).

Lemma 4.5.15 ((Malbon, 2025, Lemma 7))**.** *Let H be a general hyperplane section on X. Then the linear system $|2H - C_4|$ gives rise to a G-birational involution $\varphi : X \dashrightarrow X$. There exists a G-equivariant commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \widetilde{\varphi}\ } & \widetilde{X} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \underset{\varphi}{\dashrightarrow} & X \end{array}$$

where $\pi : \widetilde{X} \to X$ is the blowup of C_4, $\widetilde{R}$ is the strict transform on $\widetilde{X}$ of the surface R, and $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{X})$ has order 2. Let E be the exceptional divisor of π. Then $\widetilde{\varphi}(E) = \widetilde{R}$. Moreover, one has $\widetilde{R} \simeq E \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Note that $\widetilde{X}$ above is a smooth Fano threefold of Picard rank 2 and degree 28. More details about $\widetilde{X}$ can be found in Malbon (2024) or (Araujo et al., 2023a, Section 5.9).

Remark 4.5.16. Recall that there is an involution $\sigma \in \mathrm{Aut}(X)$ such that $\langle \sigma, G \rangle \simeq \mathfrak{S}_5$. In many cases, G-orbits or G-invariant curves appear in pairs swapped by σ. For example, σ swaps C_4 and C_4'. By symmetry, $|2H - C_4'|$ gives an involution φ' similar to φ. We construct φ and φ' in equations and find that $\varphi(C_4')$ and $\varphi'(C_4)$ are two smooth irreducible curves of degree 8. Each curve is cut out by cubics passing through it. We also find that $\varphi'(\varphi(C_4'))$ and $\varphi(\varphi'(C_4))$ are irreducible curves of degree 16 such that each of them has 12 cusps. Equations of φ, φ' and the curves can be found in Pinardin and Zhang (2025a).

Lemma 4.5.17. *Let C be a G-invariant curve of degree at most 17 contained in R or R' and $C \neq C_4, C_4'$. Then each irreducible component of C is not a center of non-canonical singularities of $(X, \lambda \mathscr{M}_X)$.*

Proof. Without loss of generality, we may assume that $C \subset R$. Suppose that $\mathrm{mult}_C(\lambda \mathscr{M}_X) > 1$. Let us seek for a contradiction. We use the notation of Lemma 4.5.15. Set $\widetilde{H} = \pi^*(H)$, let $\mathscr{M}_{\widetilde{X}}$ be the strict transform on $\widetilde{X}$ of the linear system $\mathscr{M}_X$, and $\widetilde{C}$ the strict transform on $\widetilde{X}$ of the curve C. Then

$$\widetilde{R} \sim 2\widetilde{H} - 3E, \quad \widetilde{H} \cdot \widetilde{C} \leq 17, \quad \widetilde{C} \not\subset E, \quad \text{and} \quad \mathrm{mult}_{\widetilde{C}}(\lambda \mathscr{M}_{\widetilde{X}}) > 1,$$

where

$$\lambda \mathscr{M}_{\widetilde{X}} \sim_{\mathbb{Q}} 3\widetilde{H} - rE$$

for some $r \in \mathbb{Q}_{\geqslant 0}$. By Lemma 4.5.15, there is an involution $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{X})$ such that $\widetilde{\varphi}(\widetilde{R}) = E$ and

$$\widetilde{\varphi}^*(\widetilde{H}) \sim 2\widetilde{H} - E.$$

In particular, we know that $\widetilde{R} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, we have $\widetilde{R}|_E = 2\Delta$, where Δ is a smooth curve in E of bidegree $(1,1)$. This implies that the G-action on E is diagonal.

To obtain a contradiction, we consider the restriction

$$\lambda \mathscr{M}_{\widetilde{X}}|_{\widetilde{R}} \sim_{\mathbb{Q}} (3\widetilde{H} - rE)|_{\widetilde{R}}$$

and show that the inequality $\mathrm{mult}_{\widetilde{C}}(\lambda \mathscr{M}_{\widetilde{X}}|_{\widetilde{R}}) > 1$ contradicts

$$\widetilde{H} \cdot \widetilde{C} \leq 17,$$

since $\widetilde{C} \neq \Delta$. To do this, we restrict $\widetilde{\varphi}(\mathscr{M}_{\widetilde{X}})$ to E.

Namely, set $\mathscr{M}_{\widetilde{X}}' = \widetilde{\varphi}(\mathscr{M}_{\widetilde{X}})$ and $\widetilde{C}' = \widetilde{\varphi}(\widetilde{C})$. Then $(2\widetilde{H} - E) \cdot \widetilde{C}' \leq 17$, $\widetilde{C}' \neq \Delta$ and

$$\mathrm{mult}_{\widetilde{C}'}(\lambda \mathscr{M}_{\widetilde{X}}') > 1,$$

where

$$\lambda \mathscr{M}'_{\widetilde{X}} \sim_{\mathbb{Q}} 3(2\widetilde{H} - E) - r\widetilde{R}.$$

Let f be a fibre of the natural projection $\pi|_E : E \to C_4$, and s a section of this projection such that $s^2 = 0$. Then

$$\widetilde{C}' \sim as + bf$$

for some non-negative integers a and b. We have

$$(as + bf) \cdot (2\widetilde{H} - E)|_E = (2\widetilde{H} - E) \cdot \widetilde{C}' \leq 17. \tag{18}$$

We compute $2\widetilde{H}|_E \sim 8f$ and $E|_E = s - 5f$, so $(2\widetilde{H} - E)|_E \sim s + 3f$. Plugging this into (18), we get

$$3a + b \leq 17.$$

Since $\widetilde{C}' \neq \Delta$, we know that $a \neq b$. A computation of G-invariant forms on E then implies that

$$(a, b) \in \{(0, 12), (1, 11), (1, 13), (2, 10)\}. \tag{19}$$

Now, we use the inequality $m := \mathrm{mult}_{\widetilde{C}'}(\lambda \mathscr{M}'_{\widetilde{X}}) > 1$. It gives

$$\lambda \mathscr{M}'_{\widetilde{X}}|_E = m\widetilde{C}' + \Omega,$$

where Ω is a $\mathbb{Q}$-linear system on E. On the other hand, we have

$$\lambda \mathscr{M}'_{\widetilde{X}}|_E \sim_{\mathbb{Q}} 3(s + 3f) - 2r(s + f) = (3 - 2r)s + (9 - 2r)f,$$

and thus $m(as + bf) + \Omega \sim_{\mathbb{Q}} (3 - 2r)s + (9 - 2r)f$. This yields

$$b < bm \leqslant s \cdot (m(as + bf) + \Omega) = s \cdot ((3 - 2r)s + (9 - 2r)f) = 9 - 2r \leqslant 9,$$

which contradicts (19). This completes the proof. □

Using the geometry of $\widetilde{X}$, we can show that C_8 and C_8' described in Remark 4.5.16 are the only G-invariant rational curves of degree 8 in X.

Lemma 4.5.18. *Let C be a G-invariant curve of degree 4 in X. Then $C = C_4$ or C_4'. Let C be a G-invariant rational curve of degree 8 in X. Then $C = C_8$ or C_8', where $C_8 = \varphi(C_4')$ and $C_8' = \varphi'(C_4)$.*

Proof. By Lemma 4.5.9/(1), any G-invariant curve C of degree 4 is contained in R or R'. Without loss of generalities, assume $C \subset R$. The first assertion then follows from the proof of Lemma 4.5.17. In particular, no solution in (19) gives $3a + b = 4$. To show the second assertion, recall that any G-invariant curve of degree 8 is irreducible since 8 is not a multiple of the index of any strict subgroup of G. By the proof of Lemma 4.5.9/(1), we know that C contains an orbit of length 12. Assume that $\Sigma_{12} \subset C$. Under the same notation as in the proof of Lemma 4.5.17, let $\widetilde{C}$ be the strict transform of C in $\widetilde{X}$. We have

$$\widetilde{C} \cdot (2\widetilde{H} - E) = 2 \cdot 8 - 12 = 4.$$

It follows that $\varphi(C)$ is a curve of degree 4 in R', which is necessarily C_4'. Then $C = \varphi(C_4')$ since φ is an involution. Similarly, we can show that $C = \varphi'(C_4)$ when $\Sigma_{12}' \in C$, □

Lemma 4.5.19. *Let C be a singular G-invariant curve of degree 16 that is not contained in $R \cup R'$. Then $C = \varphi'(C_8)$ or $C = \varphi(C_8')$. Moreover, C is not a non-canonical center of the log pair $(X, \lambda \mathscr{M}_X)$.*

Proof. First, we show that C is singular along an orbit of length 12. Assume it is not. Then $|\mathrm{Sing}(C)| \geq 20$. Let S be the unique smooth K3 surface in $\mathscr{P}$ containing C as in Lemma 4.5.9, and H_S a general hyperplane section on S. We have $C^2 \leq 42$ by the Hodge index theorem, and thus the arithmetic genus $p_a(C) \leq 22$. Since $|\mathrm{Sing}(C)| \geq 20$, the geometric genus $g(C) \leq 2$. By (Cheltsov & Shramov, 2016b, Lemma 5.1.5.), $\mathfrak{A}_5$ does not act on curves of geometric genus 1 or 2. It follows that

$$g(C) = 0, \quad |\mathrm{Sing}(C)| = p_a(C) = 20, \quad C^2 = 38.$$

Then $(C - 2H_S)^2 = -2$. This implies that $|C - 2H_S|$ is not empty and has a fixed component of degree ≤ 4, which is impossible by Lemma 4.5.9. Thus, C is singular along an orbit of length 12. Assume that C is singular at points in Σ_{12}. Under the same notation as in the proof of Lemma 4.5.18, we have that

$$\deg(\varphi(C)) = \widetilde{C} \cdot (2\widetilde{H} - E) = 2 \cdot 16 - 2 \cdot 12 - (a \cdot 12 + b \cdot 20 + c \cdot 30 + d \cdot 60) \geq 0$$

for $a, b, c, d \in \mathbb{Z}_{\geq 0}$. Then the only possibility is $\deg(\varphi(C)) = 8$. It follows from Lemma 4.5.18 that $\varphi(C) = C_8$ or C_8', i.e., $C = \varphi(C_8)$ or $\varphi(C_8')$. The first is impossible since $\varphi(C_8) = C_4'$ has degree 4. Thus, $C = \varphi(C_8')$. Similarly, if C is singular at points in Σ_{12}', we can show that $C = \varphi'(C_8)$.

Then we can find equations of C. We listed them in Pinardin and Zhang (2025a). Using equations, we check that $|\mathrm{Sing}(C)| = 12$ and $|5H_S - C|$ contains no base curve other than C. Then $5H_S - C$ is nef. Since C_8 and C_8' are rational curves, we know $g(C) = 0$. It follows that $p_a(C) = 12$ and $C^2 = 22$.

Now assume that C is a non-canonical center of $(X, \lambda \mathscr{M}_X)$. Similarly as in the proof of Lemma 4.5.12, we know that $3H_S - C$ is effective. But computing

$$(5H_S - C) \cdot (3H_S - C) = -16 < 0,$$

we obtain a contradiction to the nefness of $5H_S - C$. This completes the proof.

□

4.5.3 Orbits of points

Now we study when all non-canonical centers are points. First, we show that points in the invariant curves of degree 4 or 8 cannot be non-canonical centers in this case.

Lemma 4.5.20. *Suppose that C_4 is not a non-canonical center of the log pair $(X, \lambda \mathscr{M}_X)$, then every point in C_4 is not a non-canonical center. The same holds for C_4'.*

Proof. Let $a = \mathrm{mult}_{C_4}(\lambda \mathscr{M}_X)$. By assumption, we have $a \leq 1$. We retain the notation in Lemma 4.5.17: let $\pi : \widetilde{X} \to X$ be the blowup of C_4, and E its exceptional divisor, so that $E = \mathbb{P}^1 \times \mathbb{P}^1 \to C_4$. Let $\lambda \mathscr{M}_{\widetilde{X}}$ be the linear system satisfying

$$K_{\widetilde{X}} + \lambda \mathscr{M}_{\widetilde{X}} + (a-1)E \sim \varphi^*(K_X + \lambda M_X).$$

Assume that a point on C_4 is a non-canonical center of $(X, \lambda \mathscr{M}_X)$. Then there exists a center Z of non-canonical singularities of the pair

$$(\widetilde{X}, \lambda \mathscr{M}_{\widetilde{X}} + (a-1)E)$$

such that $Z \subset E$. It follows that Z is a center of non-canonical singularities of $(\widetilde{X}, \lambda \mathscr{M}_{\widetilde{X}} + E)$. By inversion of adjunction, Z is a non-log-canonical center of $(E, \lambda \mathscr{M}_{\widetilde{X}}|_E)$. Note that $\lambda \mathscr{M}_{\widetilde{X}}|_E$ is not mobile, and consider a divisor $D \sim_{\mathbb{Q}} \lambda \mathscr{M}_{\widetilde{X}}|_E$. Let f be a general fibre of $E \to C_4$ and s a section such that $s^2 = 0$. We compute

$$D \sim_{\mathbb{Q}} as + (12 - 5a)f \in \mathrm{Pic}(E) \otimes \mathbb{Q}.$$

Since $a \leq 1$, we know that (E, D) is log-canonical at a general point of any curve which is not a fibre. On the other hand, if any fibre is a non-log-canonical center of (E, D), then at least 12 fibres are non-log-canonical centers, since the smallest orbit of the $\mathfrak{A}_5$-action on $C = \mathbb{P}^1$ has length 12. This is impossible because $12 - 5a \leq 12$. It follows that (E, D) is not log-canonical at finitely many points. Let p be one of these points and L a fibre containing p. Write

$$D \sim_{\mathbb{Q}} bL + \Delta, \quad b \leq 1$$

for some divisor Δ not supported along L. Then $(E, L + \Delta)$ is also not log-canonical at p. By inversion of adjunction, $(L, \Delta|_L)$ is not log-canonical at p, which contradicts

$$(\Delta \cdot L)_p = a \leq 1.$$

□

Lemma 4.5.21. *Suppose that the curves C_4, C_4', C_8 and C_8' are not centers of non-canonical singularities of $(X, \lambda\mathscr{M}_X)$, then none of the points in $C_4 \cup C_4' \cup C_8 \cup C_8'$ is a non-canonical center.*

Proof. By Lemma 4.5.20, it suffices to show that $(X, \lambda\mathscr{M}_X)$ is canonical at any point in C_8 under the assumption. Let Σ be the intersection of C_8 with the non-log-canonical locus of $(X, \lambda\mathscr{M}_X)$. Similarly as before, we may assume that Σ is 0-dimensional. Let M_1, M_2 be two general members in $\mathscr{M}_X$, and write

$$\lambda^2(M_1 \cdot M_2) = mC_8 + \Delta$$

for some divisor Δ not supported along C_8 and $m \geq 0$. Intersecting with a general hyperplane H, we obtain

$$18 = \lambda^2(M_1 \cdot M_2 \cdot H) \geq m\deg(C_8) = 8m \quad \Rightarrow \quad m \leq 9/4.$$

Recall that C_8 is cut out by cubics, see Remark 4.5.16. Let S be a general cubic on X passing through C_8. Since C_8 is smooth, by Theorem 1.6.8, we have

$$\mathrm{mult}_\Sigma\left(\lambda^2(M_1 \cdot M_2)\right) > 4 \quad \Rightarrow \quad \mathrm{mult}_\Sigma(\Delta) \geq 4 - m.$$

Observe that

$$54 = \lambda^2(M_1 \cdot M_2 \cdot S) = 24m + \Delta \cdot S \geq 24m + |\Sigma|(4 - m). \tag{20}$$

Since $4 - m > 0$, the inequality (20) implies that $|\Sigma| < 20$. By Lemma 4.5.3, Σ consists of orbits of length 5 or 12. Orbits of length 12 are in $C_4 \cup C_4'$, and thus are excluded by Lemma 4.5.20. On the other hand, none of orbits of length 5 is in C_8. It follows that $\Sigma = \emptyset$. □

Proposition 4.5.22. *Suppose that the log pair $(X, \lambda\mathscr{M}_X)$ is canonical away from finitely many points. Then it is canonical away from $\Sigma_5 \cup \Sigma_5'$.*

Proof. Let Σ be the non-canonical locus of $(X, \lambda\mathscr{M}_X)$. By Remark 1.6.11, $(X, \frac{3}{2}\lambda\mathscr{M}_X)$ is not log-canonical at points in Σ. Let ε be a positive rational number such that

$$\Sigma \subset \Omega, \quad \Omega := \mathrm{Nklt}(X, (\frac{3}{2} - \varepsilon)\lambda\mathscr{M}_X).$$

Assume that Ω contains some curve C. Let

$$m = \mathrm{mult}_C((\frac{3}{2} - \varepsilon)\lambda\mathscr{M}_X).$$

Observe that

$$1 < m < \frac{3}{2} \quad \Rightarrow \quad \frac{m^2}{m-1} > \frac{9}{2}.$$

Let $M_1, M_2 \in \mathscr{M}_X$ be two general elements, and H a general hyperplane on X. By Theorem 1.6.10, we know that

$$\frac{9}{2}\deg(C) < \frac{m^2}{m-1}\deg(C) \leq (\frac{3}{2}-\varepsilon)^2\lambda^2(H\cdot M_1\cdot M_2) < \frac{9}{4}18 = \frac{81}{2},$$

which implies that $\deg(C) \leq 8$. It follows that C can only be one of C_4, C_4', C_8 and C_8'. By Lemma 4.5.20 and Lemma 4.5.21, Σ is disjoint from these curves. Thus, the 0-dimensional component Ω_0 of Ω is non-empty. Applying Nadel vanishing in the same way as in the proof of Proposition 4.3.13, we obtain that $|\Sigma| \leq |\Omega_0| < 14$. Since all orbits of length 12 are contained in C_4 or C_4', the proof is complete. □

4.6 The nonstandard $\mathfrak{A}_5$-action on the cubic threefold

Now, we focus on the other model of X: a cubic threefold Y with $5\mathrm{A}_2$-singularities, carrying the same G-action generated by (9). Let f_1 and f_2 be the cubics defined in (10). Then Y is given by

$$Y = \{(8-3\zeta_6)f_1 + 7f_2 = 0\} \subset \mathbb{P}^4 \tag{21}$$

with the same G-action given by (9). The aim of this section is to prove the following result.

Proposition 4.6.1. *Let $\mathscr{M}_Y$ be a non-empty mobile G-invariant linear system on Y, and $\mu \in \mathbb{Q}$ such that $\mu\mathscr{M}_Y \sim_{\mathbb{Q}} -K_Y$. Then the log pair $(Y, \mu\mathscr{M}_Y)$ is canonical away from $\mathrm{Sing}(Y)$.*

Proof. This follows from Propositions 4.6.11 and 4.6.12. □

First, we classify small orbits and curves of degrees at most 11. We show that all such curves are reducible. In the second subsection, we study singularities of pairs $(Y, \mu\mathscr{M}_Y)$ as above along invariant curves, and in the third subsection, we consider 0-dimensional centers.

4.6.1 Small G-orbits and G-invariant curves of low degrees

Lemma 4.6.2. *A G-orbit of points in Y with length ≤ 20 is one of the following:*

$$\begin{aligned}
\Sigma_5 &= \textit{the orbit of} \quad [1:\zeta_6-1:-\zeta_6:\zeta_6-1:1],\\
\Sigma_{12} &= \textit{the orbit of} \quad [\zeta_5^3:\zeta_5^2:0:\zeta_5:1],\\
\Sigma_{12}' &= \textit{the orbit of} \quad [\zeta_5^4:\zeta_5:0:\zeta_5^3:1],\\
\Sigma_{15} &= \textit{the orbit of} \quad [1:0:0:0:0],\\
\Sigma_{20} &= \textit{the orbit of} \quad [(3\zeta_6-8):(-8\zeta_6+5):(5\zeta_6+3):7(\zeta_6-1):7].
\end{aligned}$$

where the length of each orbit is indicated by the subscript. A G-orbit of points in Y with length 30 is the orbit of a general point in one of the following two curves:

$$\text{a cuspidal cubic curve:} \quad \{x_1 - x_4 = x_2 - x_3 = 0\} \cap Y,$$

or

$$\text{a line:} \quad \{x_1 + x_4 = x_2 + x_3 = x_5 = 0\} \subset Y.$$

Moreover, every G-orbit of points in Y of length different from 60 is one of the orbits described above.

Observe that Σ_5 is the singular locus of Y. We now describe the G-invariant reducible curves of degrees lower than 12 on Y. Later, we show that they are the only G-invariant curves of such degrees.

Lemma 4.6.3. *Let C be a G-invariant reducible curve of degree at most 11. Then C is the union of curves in one of the following orbits:*

- *one of the following two orbits of 6 lines*

$$\mathscr{L}_6 = \text{the orbit of } \{x_1 + x_4 + (-\zeta_5^3 - \zeta_5^2)x_5 = x_2 + (\zeta_5^3 + \zeta_5^2)x_4 + \\ + (\zeta_5^3 + \zeta_5^2)x_5 = x_3 - (\zeta_5^3 + \zeta_5^2)x_4 + x_5 = 0\},$$

$$\mathscr{L}_6' = \text{the orbit of } \{x_2 - (\zeta_5^3 + \zeta_5^2 + 1)x_4 - (\zeta_5^3 + \zeta_5^2 + 1)x_5 = x_1 + \\ + x_4 + (\zeta_5^3 + \zeta_5^2 + 1)x_5 = x_3 + (\zeta_5^3 + \zeta_5^2 + 1)x_4 + x_5 = 0\},$$

- *one of the following two orbits of 10 lines*

$$\mathscr{L}_{10} = \text{the orbit of } \{x_1 + x_3 + x_5 = (5\zeta_3 - 3)x_4 + 7x_5 = \\ = 7x_1 + (5\zeta_3 - 3)x_2 = 0\},$$

$$\mathscr{L}_{10}' = \text{the orbit of } \{x_1 + x_3 + x_5 = -\zeta_3 x_4 + x_5 = x_1 - \zeta_3 x_2 = 0\}.$$

Moreover, $\mathscr{L}_{10}'$ consists of ten lines passing through pairs of 5 singular points of Y. The lines in $\mathscr{L}$ are pairwise disjoint for $\mathscr{L} = \mathscr{L}_6, \mathscr{L}_6'$ or $\mathscr{L}_{10}$.

Proof. The proof is similar to that of Lemma 4.3.5. □

The rest of this subsection is devoted to proving the following result.

Proposition 4.6.4. *Let C be a G-invariant curve in Y of degree at most 11. Then C is the union of all lines in one of the orbits $\mathscr{L}_6$, $\mathscr{L}_6'$, $\mathscr{L}_{10}$, or $\mathscr{L}_{10}'$ given in Lemma 4.6.3.*

Proof. This follows from Lemmas 4.6.7, 4.6.8, 4.6.9, and 4.6.10. □

First we notice that there is a unique G-invariant surface in the linear system $|\mathscr{O}_Y(2)|$ and $|\mathscr{O}_Y(3)|$. We denote them by Q and R respectively. We have

$$\Sigma_{12}, \Sigma'_{12} \in Q \cap R, \quad \Sigma_5 \in R \setminus Q, \quad \Sigma_{20} \in Q \setminus R. \tag{22}$$

By computation, we find that Q is a nodal K3 surface.

Lemma 4.6.5. *The singular locus of Q is Σ_5 and each singular point is an ordinary double point. The singular locus of R is $\Sigma_{12} \cup \Sigma'_{12}$ and each singular point is an ordinary double point.*

Lemma 4.6.6. *Let $C \subset Y$ be an irreducible G-invariant curve of degree at most 11 which is not a union of ten lines. Then $\Sigma_5 \not\subset C$.*

Proof. Assume that $\Sigma_5 \subset C$. Let $\widetilde{Y}$ be the blowup of Y in Σ_5, let E be the exceptional divisor, and $\widetilde{C}$ be the strict transform of C. We denote by H the pullback to $\widetilde{Y}$ of a general hyperplane section on Y. The base locus of the linear system $|4H - 3E|$ is the strict transform of the union of the ten lines that pass through pairs of points in Σ_5. By assumption, the curve $\widetilde{C}$ is not in the base locus of $|4H - 3E|$, so we have $(4H - 3E) \cdot \widetilde{C} \geq 0$. On the other hand,

$$0 \leq (4H - 3E) \cdot \widetilde{C} = 4d - 15E_1 \cdot \widetilde{C} \leq 44 - 15E_1 \cdot \widetilde{C} \tag{23}$$

where E_1 is an irreducible component of E. The divisor E_1 is a quadric cone and is invariant under an $\mathfrak{A}_4$-action. Then $\mathfrak{A}_4$ acts faithfully on the base conic of the cone. The orbit of a smooth point in the cone is at least of length 4. It follows that if $\widetilde{C}$ does not pass through the vertex P of E_1, we have $E_1 \cdot \widetilde{C} \geq 4$, contradicting (23). If $\widetilde{C}$ passes through P, it is singular at P, because otherwise $\mathfrak{A}_4$ does not act faithfully on the tangent space of $\widetilde{C}$ at P. Hence $E_1 \cdot \widetilde{C} \geq 2 \cdot 2 = 4$, again contradicting (23). □

Lemma 4.6.7. *Let $C \subset Y$ be an irreducible G-invariant curve of degree $d \leq 11$. Then $d \in \{6, 8, 10\}$. Moreover, if $d = 6$ (resp. $d = 8$), then $C \subset R$ (resp. $C \subset Q$).*

Proof. By Lemma 4.6.6, $\Sigma_5 \not\subset C$. If $C \not\subset Q$, it follows from possible lengths of orbits that

$$Q \cdot C = 2d = 12a + 20b + 30c, \quad \text{for } a, b, c \in \mathbb{Z}_{\geq 0}.$$

Since $d \leq 11$, we find that

$$(d, a, b, c) \in \{(6, 1, 0, 0), (10, 0, 1, 0)\}.$$

Similarly, if $C \not\subset R$, we have

$$R \cdot C = 3d = 12a + 20b + 30c, \quad \text{for } a, b, c \in \mathbb{Z}_{\geq 0}.$$

Recall that R is singular at the orbits of length 12 by Lemma 4.6.5. Then $a = 0$ or $a \geq 2$. The only possibilities are

$$(d,a,b,c) \in \{(8,2,0,0),(10,0,0,1)\}.$$

Therefore, when $d \notin \{6,8,10\}$, we know that $C \subset Q \cap R$. But $Q \cap R$ is an irreducible curve of degree 18. □

Next, we show that there is also no irreducible G-invariant curve of degree 6, 8 or 10.

Lemma 4.6.8. *Let C be a G-invariant curve of degree 6 in Y. Then C is a union of 6 lines in $\mathscr{L}_6$ or $\mathscr{L}_6'$ given in Lemma 4.6.3.*

Proof. Assume that C is not a union of 6 lines. By Lemma 4.6.3, C is irreducible. Lemma 4.6.7 shows that $C \subset R$. From (22), we know that Q contains both orbits of length 12. Since $C \cdot Q = 12$, the curve C must contain one and only one orbit of length 12 and C is smooth along this orbit. Assume that C contains Σ_{12}. Recall from Lemma 4.6.5 that points in Σ_{12} are nodes of R. Let $f\colon \widetilde{R} \to R$ be the blowup of R at Σ_{12}, E its exceptional divisor, and $\widetilde{C}$ the strict transform of C. Then

$$\widetilde{C} \sim_{\mathbb{Q}} f^*(C) - \frac{a}{2}E,$$

for some positive integer a. By Hodge index theorem, we have $C^2 \leq 4$. It follows that

$$\begin{aligned} 2p_a(\widetilde{C}) - 2 &= (K_{\widetilde{R}} + \widetilde{C}) \cdot \widetilde{C} \\ &= (f^*(H) + f^*(C) - \frac{a}{2}E) \cdot \widetilde{C} \\ &= 6 + C^2 - \frac{12a^2}{2} \\ &\leq 10 - 6a^2. \end{aligned}$$

We deduce that $p_a(\widetilde{C}) \leq 6 - 3a^2 \leq 3$. Since there is no G-orbits of length ≤ 3, the geometric genus $g(\widetilde{C}) = p_a(\widetilde{C})$, i.e., both $\widetilde{C}$ and C are smooth and $g(C) \leq 3$. From (Cheltsov & Shramov, 2016b, Lemma 5.1.5), we know that C is a smooth rational curve. Then C contains a G-orbit of length 20. On the other hand, the only G-orbit of length 20 is contained in Q by (22). This contradicts $C \cdot Q = 12$. □

Lemma 4.6.9. *There is no G-invariant curve of degree 8 in Y.*

Proof. Assume that C is such a curve. It is necessarily irreducible. Lemma 4.6.7 shows that $C \subset Q$. Recall from Lemma 4.6.5 that Q is a K3 surface singular at Σ_5. Hence, by (Cheltsov & Shramov, 2016a, Proposition 6.7.3), we have $\mathrm{Pic}^G(Q) \cong \mathbb{Z}$. Let H be a general hyperplane section on Q. Since $H^2 = 6$ is not a square, we know that H generates $\mathrm{Pic}^G(Q)$. On the other hand, Lemma 4.6.6 implies that C is contained in the smooth locus of Q, and thus is a Cartier divisor. It follows that $C = nH$, for some $n \in \mathbb{Z}$. Then $8 = C \cdot H = 6n$, which is impossible. □

Lemma 4.6.10. *Let C be a G-invariant curve of degree 10 in Y. Then C is a union of 10 lines in $\mathscr{L}_{10}$ or $\mathscr{L}'_{10}$ given in Lemma 4.6.3.*

Proof. Assume that C is not a union of 10 lines. By Lemma 4.6.3, C is irreducible. Consider its normalization $f\colon C' \to C$. Since C is not contained in $R \cap Q$, the proof of Lemma 4.6.7 shows that C cannot have an orbit of length 12. By (Cheltsov & Shramov, 2016b, Lemma 5.1.5), we deduce that the genus of C' satisfies

$$g(C') \geq 6.$$

Consider the divisor $D = f^*(\mathscr{O}_C(3))$ on C', and the restriction map

$$H^0(Y, \mathscr{O}_Y(3)) \longrightarrow H^0(C, \mathscr{O}_C(3)),$$

We want to estimate $h^0(D) = h^0(C', \mathscr{O}_{C'}(D))$. If D is non-special, then by Riemann–Roch:

$$h^0(D) = \deg D - g + 1 = 30 - g + 1 \leq 25.$$

If D is special, then by the Clifford theorem:

$$h^0(D) \leq \frac{\deg D}{2} + 1 = 16.$$

Consider the map $g : H^0(Y, \mathscr{O}_Y(3)) \to H^0(C', \mathscr{O}_{C'}(D))$ given as the composition of

$$H^0(Y, \mathscr{O}_Y(3)) \xrightarrow{|_C} H^0(C, \mathscr{O}_C(3)) \xrightarrow{f^*} H^0\left(C', \mathscr{O}_{C'}(D)\right).$$

Since $h^0(Y, \mathscr{O}_Y(3)) = 34$, we find that the kernel of this map has dimension at least

$$34 - h^0(D) \geq 9.$$

This kernel consists of cubic hypersurfaces in Y that contain C.

Arguing in the same way as in Lemma 4.6.9, we see that C cannot be contained in Q. Then $C \cdot Q = 20$, so that C must contain Σ_{20}. Let V_{15} and V_{20} be the subspaces of $H^0(Y, \mathscr{O}_Y(3))$ consisting of cubics containing the orbit Σ_{15} and Σ_{20} respectively. If $\Sigma_{15} \subset C$, then $\ker \varphi \subset V_{15} \cap V_{20}$. But we compute that $\dim(V_{15} \cap V_{20}) = 5 < 9$. Hence, the curve C cannot contain Σ_{15}.

Since $R \cdot C = 30$, the curve C must contain an orbit of length 30 lying on R. Let Σ_{30} be such an orbit, and V_{30} the space of cubics containing Σ_{30}. We have explicitly described these orbits in Lemma 4.6.2. In particular, Σ_{30} is either the orbit of one of the following 4 points

$$[-\zeta_4 - 2 : 1 : 1 : 1 : 1], \quad [\zeta_4 - 2 : 1 : 1 : 1 : 1],$$

$$[-2\zeta_5^3 - 2\zeta_5^2 : 1 : \zeta_5^3 + \zeta_5^2 - 1 : \zeta_5^3 + \zeta_5^2 - 1 : 1],$$

$$[2\zeta_5^3 + 2\zeta_5^2 + 2 : 1 : -\zeta_5^3 - \zeta_5^2 - 2 : -\zeta_5^3 - \zeta_5^2 - 2 : 1],$$

or the orbit of a general point in the line

$$\{x_1 + x_4 = x_2 + x_3 = x_5 = 0\} \subset Y.$$

A linear algebra computation then shows that $\dim(V_{20} \cap V_{30}) < 9$ for any such orbit Σ_{30}. Therefore we obtain a contradiction. □

4.6.2 Invariant curves

Proposition 4.6.11. *Let C be a G-invariant curve in Y. Then each irreducible component of C is not a non-canonical center of $(Y, \mu\mathscr{M}_Y)$.*

Proof. Assume that irreducible components of C are non-canonical centers of $(Y, \mu\mathscr{M}_Y)$. Proposition 4.6.4 shows that C is the union of all curves in one of the following orbits given in Lemma 4.6.3

$$\mathscr{L}_6, \quad \mathscr{L}_6', \quad \mathscr{L}_{10}, \quad \text{or} \quad \mathscr{L}_{10}'.$$

Assume that C is one of the unions of six lines in $\mathscr{L}_6$ or $\mathscr{L}_6'$ which are pairwise disjoint. Let $\widetilde{Y} \to Y$ be the blowup of C in Y, E the exceptional divisor, and H the pullback of a general hyperplane section on Y. One can check that C is cut out by cubics, which implies that $|3H - E|$ is nef. By our assumption that irreducible components of C are non-canonical centers of $(Y, \mu\mathscr{M}_Y)$, we know that $|2H - mE|$ is mobile for some $m > 1$, and thus $|2H - E|$ is mobile as well. It follows that the divisor $(2H - E)^2$ is effective. On the other hand, we have $(3H - E) \cdot (2H - E)^2 = -6 < 0$, which is a contradiction.

If C is the union of 10 lines in $\mathscr{L}_{10}$ which are pairwise disjoint, we proceed in exactly the same way as C is also cut out by cubics, so $(4H - E)$ is nef and $(4H - E) \cdot (2H - E)^2 = -32 < 0$. Finally, the case where C is the union of curves in $\mathscr{L}_{10}'$, i.e., the union of the ten lines passing through pairs of points in Σ_5, is excluded the same way as in (Cheltsov, Sarikyan, & Zhuang, 2023, Proof of Proposition 3.4). □

4.6.3 Orbits of points

Proposition 4.6.12. *Let $P \in Y$ be a point and Σ its G-orbit. If P is a non-canonical center of $(Y, \mu\mathscr{M}_Y)$, then $\Sigma = \Sigma_5$.*

Proof. Assume that P is a non-canonical center. By Remark 1.6.11, we know that P is a non-log-canonical center of $(Y, \frac{3}{2}\mu\mathscr{M}_Y)$. Let ε be a positive rational number such that

$$\Sigma \subset \Omega, \quad \Omega := \mathrm{Nklt}(Y, (\frac{3}{2} - \varepsilon)\mu\mathscr{M}_Y).$$

Assume that Ω contains a curve C. Let

$$m = \mathrm{mult}_C((\frac{3}{2}-\varepsilon)\lambda\mathscr{M}_X).$$

Observe that

$$1 < m < \frac{3}{2} \quad \Rightarrow \quad \frac{m^2}{m-1} > \frac{9}{2}.$$

Let $M_1, M_2 \in \mathscr{M}_X$ be two general elements and H a general hyperplane section of Y. Then by Theorem 1.6.10, we have that

$$27 \geq (\frac{3}{2}-\varepsilon)^2\mu^2(M_1 \cdot M_2 \cdot H) \geq \frac{m^2}{m-1}\deg(C) > \frac{9}{2}\deg(C),$$

which implies that $\deg(C) < 6$. Proposition 4.6.4 shows that such curves do not exist. It follows that Ω contains no curve.

Now notice that

$$K_Y + (\frac{3}{2}-\varepsilon)\mu\mathscr{M}_Y + 2\varepsilon\mathscr{O}_Y(1) \sim_{\mathbb{Q}} \mathscr{O}_Y(1).$$

Let $\mathscr{I}$ be the multiplier ideal sheaf of $(\frac{3}{2}-\varepsilon)\mu\mathscr{M}_Y$. By Nadel vanishing theorem, we have

$$|\Omega| \leq h^0(\mathscr{O}_Y(1)) = 5.$$

It follows that $\Omega = \Sigma = \Sigma_5$. □

4.7 Proof of Theorems 4.1.1, 4.1.2 and 4.1.3

In this section, we explain how the results in Sections 4.3 to 4.6 prove Theorems 4.1.1 and 4.1.2, and can be adapted to show Theorem 4.1.3 about the nonstandard $\mathfrak{S}_5$-action. For the standard $\mathfrak{A}_5$-action, the result readily follows from Cheltsov, Sarikyan, and Zhuang (2023).

Proof of Theorem 4.1.1. By (Cheltsov, Sarikyan, & Zhuang, 2023, Section 3), this follows from Proposition 4.3.1 and 4.4.1. □

Similarly, in the case of the nonstandard $\mathfrak{A}_5$-action, Theorem 4.1.2 follows from Propositions 4.5.1 and 4.6.1. We explain this implication in detail.

4.7.1 Nonstandard $\mathfrak{A}_5$-action

We introduce some notation first. Let X be the quadric given by (8), Y the cubic given by (21), with the nonstandard G-action given by (9). We denote by Σ_5 and Σ_5' the two orbits in X of length 5, χ and χ' the Cremona map associated with them respectively, and Σ_5^Y the orbit of length 5 in Y. Let $\mathrm{Bir}^G(X)$ be the group of G-birational automorphisms of X.

It is well-known (see e.g., (Cheltsov, Sarikyan, & Zhuang, 2023, Section 3), (Cheltsov & Shramov, 2016a, Theorem 3.3.1) and Cheltsov, Dubouloz, and Kishimoto (2023); Cheltsov and Sarikyan (2022)) that the Noether–Fano inequalities (cf. Theorem 1.6.7) imply that Theorem 4.1.2 follows from the following result.

Theorem 4.7.1. *Let $\mathscr{M}_X$ be a non-empty mobile G-invariant linear system on X and $\lambda \in \mathbb{Q}$ such that $\lambda\mathscr{M}_X \sim_{\mathbb{Q}} -K_X$. Then there exists $\gamma \in \mathrm{Bir}^G(X)$ such that either $(X, \lambda\mathscr{M}_X)$ or $(Y, \mu\mathscr{M}_Y)$ has canonical singularities, where $\mathscr{M}_Y$ is the proper transform of $\mathscr{M}_X$ by $\chi \circ \gamma$, and $\mu\mathscr{M}_Y \sim_{\mathbb{Q}} -K_Y$.*

First, let us explain why we need the birational automorphism γ in the theorem above. Recall from Section 4.5 that $(X, \lambda\mathscr{M}_X)$ can have 1-dimensional non-canonical centers. Here we show that, up to replacing $\mathscr{M}_X$ by its strict transform under a birational automorphism, $(X, \lambda\mathscr{M}_X)$ is canonical away from the orbits of length five.

Proposition 4.7.2. *Let $\mathscr{M}_X$ be a non-empty mobile G-invariant linear system. Then there exists $\gamma \in \mathrm{Bir}^G(X)$ such that $(X, \lambda'\mathscr{M}_X')$ is canonical away from $\Sigma_5 \cup \Sigma_5'$, where $\mathscr{M}_X' = \gamma_*(\mathscr{M}_X)$, and $\lambda' \in \mathbb{Q}$ such that $\lambda'\mathscr{M}_X' \sim_{\mathbb{Q}} -K_X$.*

Proof. Let $\lambda \in \mathbb{Q}$ such that $\lambda\mathscr{M}_X \sim_{\mathbb{Q}} -K_X$. If $(X, \lambda\mathscr{M}_X)$ is canonical away from $\Sigma_5 \cup \Sigma_5'$, we are done. Assume on the contrary that there exists a G-irreducible subvariety Z not contained in $\Sigma_5 \cup \Sigma_5'$, and irreducible components of Z are non-canonical centers of $(X, \lambda\mathscr{M}_X)$. Then, by Proposition 4.5.1, Z is one of the following irreducible curves:

- rational curves C_4 and C_4' of degree 4 given by (12),
- rational curves C_8 and C_8' of degree 8 described in Remark 4.5.16,
- a smooth curve C_{10} of degree 10 and genus 6.

Moreover, it follows from Lemma 4.5.13 and Lemma 4.5.15 that there exists a commutative G-equivariant diagram

$$\begin{array}{ccc} V & \overset{\chi}{\dashrightarrow} & V' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ X & \underset{\delta}{\dashrightarrow} & X \end{array}$$

where

- φ is the blowup of Z,

- χ is an biregular involution if $\deg(Z)=4$, and is a composition of flops if $\deg(Z)=8$ or 10,
- φ' is the blowup of a curve Z' with the same degree and genus as Z,
- $\delta \in \mathrm{Bir}^G(X)$.

Set $\mathscr{M}'_X = \delta_*(\mathscr{M}_X)$ and $\lambda' \in \mathbb{Q}$ such that $\lambda'\mathscr{M}'_X + K_X \sim_{\mathbb{Q}} 0$. Since $\mathrm{Pic}(X)$ is generated by $\mathscr{O}_X(1)$, we know that $\mathscr{M}_X$ is a linear subsystem of $|\mathscr{O}_X(n)|$ for $n=\frac{3}{\lambda}$. Let $n'=\frac{3}{\lambda'}$. Then $\mathscr{M}'_X \subset |\mathscr{O}_X(n')|$. We claim that $n'<n$. Indeed, let $\mathscr{M}_V$ be the strict transform of the linear system $\mathscr{M}_X$ on V. Note that $\mathrm{codim}_X(Z)=2$ and $\mathrm{mult}_Z(\lambda\mathscr{M}_X)>1$. We have that

$$0 \sim_{\mathbb{Q}} \varphi^*(K_X+\lambda\mathscr{M}_X) \sim_{\mathbb{Q}} K_V+\lambda\mathscr{M}_V+aE, \quad \text{for some } a>0.$$

Pushing forward this class to X via $\varphi\circ\chi$, we obtain that

$$K_X+\lambda\mathscr{M}'_X+aD \sim_{\mathbb{Q}} 0$$

for some effective divisor D on X. Since $K_X+\lambda'\mathscr{M}'_X \sim_{\mathbb{Q}} 0$, it follows that $\lambda'>\lambda$, i.e., $n'<n$.

To summarize, if a curve is a non-canonical center, then we can find a G-birational automorphism such that the pushforward $\mathscr{M}'_X$ is a subsystem of $|\mathscr{O}_X(n')|$ for n' strictly smaller than n. Therefore, by iterating this process, we will obtain a linear system which has no 1-dimensional non-canonical center, and thus the resulting pair is canonical away from $\Sigma_5 \cup \Sigma'_5$. □

We recall the following lemma from Abban, Cheltsov, Park, and Shramov (2024).

Lemma 4.7.3. *Let V be a threefold, $K \subset \mathrm{Aut}(V)$ a finite subgroup fixing a smooth point $P \in V$, $\mathscr{M}_V$ a non-empty mobile K-invariant linear system on V, and $\lambda \in \mathbb{Q}$ such that P is a non-canonical center of $(V,\lambda\mathscr{M}_V)$. If K acts on the Zariski tangent space $T_P(V)$ of V at P via an irreducible representation, then $\mathrm{mult}_P(\mathscr{M}_V) > \frac{2}{\lambda}$.*

Corollary 4.7.4. *Assume that $(X,\lambda\mathscr{M}_X)$ is not canonical at Σ_5. Then $\mathrm{mult}_{\Sigma_5}\mathscr{M}_X > \frac{2}{\lambda}$.*

Proof. The stabilizer of a point $P\in\Sigma_5$ is isomorphic to $\mathfrak{A}_4$, which acts on $T_P(X)$ faithfully. The only 3-dimensional faithful representation of $\mathfrak{A}_4$ is irreducible. Then we apply the previous lemma. □

Lemma 4.7.5. *Let $S=\mathbb{P}(1,1,2)$, and K a finite group acting faithfully on S such that $|\mathscr{O}_S(1)|$ has no K-invariant curves. Then $\alpha_K(S)\geq\frac{1}{2}$.*

Proof. Let L be a general element in $|\mathscr{O}_S(1)|$. Suppose $\alpha_K(S)<\frac{1}{2}$. Then there exists a K-invariant effective $\mathbb{Q}$-divisor D such that the log pair $(S,\frac{1}{2}D)$ is not log-canonical, and D satisfies

$$D \sim_{\mathbb{Q}} 4L \sim_{\mathbb{Q}} -K_S, \quad \text{and} \quad \frac{1}{2}D=\sum a_iC_i,$$

where for each i, we have $a_i \in \mathbb{Z}_{\geq 0}$, and $C_i \in |\mathscr{O}_S(d_i)|$ for some d_i. First, we show that $a_i \leq 1$ for all i. Indeed, we have $2 = \deg(\frac{1}{2}D) = \Sigma a_i d_i$. If $a_j > 1$ for some j, then $a_j = 2$ and $d_j = 1$. Since $|\mathscr{O}_S(1)|$ contains no K-invariant curve, there exists $g \in K$ such that $a_j g(C_j)$ also shows up in $\frac{1}{2}D$. This contradicts $\deg(\frac{1}{2}D) = 2$.

Let $\varepsilon \in \mathbb{Q}_{>0}$ such that $(S, \frac{1-\varepsilon}{2}D)$ is not log-canonical. Since $a_i \leq 1$ for all i, we know that $\Gamma = \mathrm{Nklt}(S, \frac{1-\varepsilon}{2}D)$ does not contain any curve. Nadel vanishing theorem then implies that Γ contains a single point, namely the vertex of S. Consider the blowup $\widetilde{S} \to S$ of the vertex and let E be the exceptional divisor. We have that

$$\widetilde{S} \cong \mathbb{F}_2, \quad K_{\widetilde{S}} \sim f^*(K_S), \quad \widetilde{D} = f^*(D) - mE, \quad \widetilde{L} = f^*(L) - \frac{1}{2}E$$

for some $m \in \mathbb{Z}_{>0}$. It follows that

$$0 \leq \widetilde{D} \cdot \widetilde{L} = 2 - m, \quad \text{and thus} \quad m\frac{1-\varepsilon}{2} < 1.$$

Since

$$f^*(K_S + \frac{1-\varepsilon}{2}D) \sim_{\mathbb{Q}} K_{\widetilde{S}} + \frac{1-\varepsilon}{2}\widetilde{D} + m\frac{1-\varepsilon}{2}E,$$

we see that $(\widetilde{S}, \frac{1-\varepsilon}{2}\widetilde{D} + m\frac{1-\varepsilon}{2}E)$ is not log-canonical at some point in E. Since $m\frac{1-\varepsilon}{2} < 1$, the pair $(\widetilde{S}, \frac{1-\varepsilon}{2}\widetilde{D} + E)$ is also not log-canonical at some point in E. By inversion of adjunction, we see that $(E, \frac{1-\varepsilon}{2}\widetilde{D}|_E)$ is not log-canonical. Note that $\widetilde{D}|_E \sim_{\mathbb{Q}} -E|_E$ is a divisor of degree $2m$ on $E = \mathbb{P}^1$. It follows that $\alpha_K(\mathbb{P}^1) < 1$. Recall from Cheltsov and Shramov (2016a) that

$$\alpha_K(\mathbb{P}^1) = \begin{cases} \frac{1}{2} & \text{if } K \cong C_n, \\ 1 & \text{if } K \cong \mathfrak{D}_n, \\ 2 & \text{if } K \cong \mathfrak{A}_4, \\ 3 & \text{if } K \cong \mathfrak{S}_4, \\ 6 & \text{if } K \cong \mathfrak{A}_5. \end{cases}$$

By our assumption that $|\mathscr{O}_S(1)|$ has no K-invariant curves, we see that K is not a cyclic group. Therefore, we obtain a contradiction and this completes the proof. □

Corollary 4.7.6. *Assume that points of Σ_5^Y are centers of non-canonical singularities of $(Y, \mu\mathscr{M}_Y)$. Consider $\pi\colon \widetilde{Y} \to Y$, the blowup of Y in Σ_5^Y. Let $m \in \mathbb{Q}$ such that $\pi^*(\mu\mathscr{M}_Y) \sim_{\mathbb{Q}} \mu\mathscr{M}_{\widetilde{Y}} + mE$, where $\mathscr{M}_{\widetilde{Y}}$ is the strict transform of $\mathscr{M}_Y$ to $\widetilde{Y}$, and E is the exceptional divisor of π. Then $m > 1$.*

Proof. Let P be a point of Σ_5, and let F be the component of E that is mapped to P. Then $F \simeq \mathbb{P}(1,1,2)$, since P is an A_2-singularity. Observe that

$$\pi^*(K_Y + \mu\mathscr{M}_Y) \sim_{\mathbb{Q}} K_{\widetilde{Y}} + \mu\mathscr{M}_{\widetilde{Y}} + (m-1)E.$$

Recall that P is a non-canonical center of $(Y, \mu \mathscr{M}_Y)$. It follows that $(\widetilde{Y}, \mu \mathscr{M}_{\widetilde{Y}} + (m-1)E)$ is not canonical at some point in F. Hence, $(\widetilde{Y}, \mu \mathscr{M}_{\widetilde{Y}} + mE)$ is not log-canonical at some point in F.

Now assume that $m \leq 1$. Then $(\widetilde{Y}, \mu \mathscr{M}_{\widetilde{Y}} + E)$ is not log-canonical at some point in F. It follows from the inversion of adjunction that $(F, \mu \mathscr{M}_{\widetilde{Y}}|_F)$ is not log-canonical. Note that

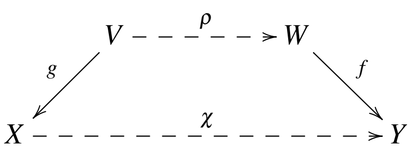
$$\mu \mathscr{M}_{\widetilde{Y}}|_F \sim_{\mathbb{Q}} -mF|_F \sim_{\mathbb{Q}} \mathscr{O}_F(2m). \tag{24}$$

The stabilizer of P is isomorphic to $\mathfrak{A}_4$, which acts faithfully on F. Then (24) implies that $\alpha_{\mathfrak{A}_4}(F) < \frac{m}{2}$, which contradicts Lemma 4.7.5. Therefore we conclude that $m > 1$. □

We are now ready to prove Theorem 4.7.1.

Proof of Theorem 4.7.1. By Proposition 4.7.2, we may assume that, up to applying a G-birational automorphism of X, the log pair $(X, \lambda \mathscr{M}_X)$ is canonical away from $\Sigma_5 \cup \Sigma_5'$. Since Σ_5 and Σ_5' are exchanged by some element in the normalizer of G in $\mathrm{Aut}(X)$, we can further assume that $(X, \lambda \mathscr{M}_X)$ is canonical away from Σ_5. Now, it suffices show that either $(X, \lambda \mathscr{M}_X)$ is canonical along Σ_5, or $(Y, \mu \mathscr{M}_Y)$ is canonical along Σ_5^Y.

We denote by H_X (resp. H_Y) a general hyperplane section on X (resp. Y). Let $n, n' \in \mathbb{Z}$ such that $\mathscr{M}_X \sim_{\mathbb{Q}} nH_X$, and $\mathscr{M}_Y \sim_{\mathbb{Q}} n'H_Y$. Note that $n = \frac{3}{\lambda}$ and $n' = \frac{2}{\mu}$. Recall from (Cheltsov, Sarikyan, & Zhuang, 2023, Section 3) that the Cremona map χ fits into the G-equivariant commutative diagram:



where g is the blowup of Σ_5, ρ is a small birational map that flops the proper transforms of 10 conics that contain three points in Σ_5, and f contracts to Σ_5^Y the proper transforms of 5 hyperplane sections of X that pass through four points in Σ_5'. Let $\widetilde{H}_X$, $\widetilde{H}_Y$, $\widetilde{\mathscr{M}_X}$, $\widetilde{\mathscr{M}_Y}$ be the strict transforms in V of H_X, H_Y, $\mathscr{M}_X$, $\mathscr{M}_Y$ respectively, E the exceptional divisor of g, and F the strict transform in V of the exceptional divisor of f. We compute

$$\widetilde{\mathscr{M}_X} = n\widetilde{H}_X - mE = (4n - 5m)\widetilde{H}_Y - (3n - 4m)F = n'\widetilde{H}_Y - m'F = \widetilde{\mathscr{M}_Y},$$

which yields

$$n' = 4n - 5m, \quad m' = 3n - 4m,$$

for some $m, m' \in \mathbb{Q}$. By Corollary 4.7.4, if $(X, \lambda \mathscr{M}_X)$ is not canonical along Σ_5, then $\lambda m > 2$, i.e., $3m > 2n$. On the other hand, if $(Y, \mu \mathscr{M}_Y)$ is not canonical at Σ_5^Y, then by Corollary 4.7.6, we have $1 < \mu m'$, i.e., $2n > 3m$. These two cases cannot happen simultaneously. We conclude that either $(X, \lambda \mathscr{M}_X)$ is canonical at Σ_5, or $(Y, \mu \mathscr{M}_Y)$ is canonical at Σ_5^Y. □

4.7.2 Nonstandard $\mathfrak{S}_5$-action

Throughout this subsection, $G' = \mathfrak{S}_5$. Using our analysis of the non-standard $\mathfrak{A}_5$-action, it is not hard to prove Theorem 4.1.3. Let X be the same quadric threefold defined by (8) and Y the cubic threefold defined by (21). We consider the *nonstandard* G'-action on X and Y generated by the nonstandard $\mathfrak{A}_5$-action (9), and an extra involution

$$\iota : (\mathbf{x}) \mapsto (x_3, x_4, x_1, x_2, -x_1 - x_2 - x_3 - x_4 - x_5).$$

We will show that the only G'-Mori fibre spaces that are G'-equivariantly birational to X are X and Y.

Recall from Propositions 4.5.6 and 4.5.22 that the possible non-canonical centers under the nonstandard $\mathfrak{A}_5$-action are points in Σ_5 and Σ_5', curves C_4, C_4', C_8, C_8', or some curves of degree 10.

Under the G'-action, the orbits Σ_5 and Σ_5' are still invariant. So, the cubic Y with 5A$_2$-singularities is still $\mathfrak{S}_5$-equivariantly birational to X. Proposition 4.6.1 clearly also holds for the $\mathfrak{S}_5$-action. We focus on the quadric. Any curve of degree 10 becomes irrelevant, since if it is not $\mathfrak{S}_5$-invariant, its $\mathfrak{S}_5$-orbit becomes a curve of degree 20, which exceeds the bound 18 as in Remark 4.3.2.

The involution $\iota \in \mathfrak{S}_5$ exchanges the curve C_4 with C_4', and C_8 with C_8'. We show that C_8 and C_8' are not non-canonical centers in this case.

Lemma 4.7.7. *The curves C_8 and C_8' are not centers of non-canonical singularities of $(X, \lambda\mathscr{M}_X)$.*

Proof. Assume C_8 is a non-canonical center. Since $\mathscr{M}_X$ is G'-invariant, C_8' is also a non-canonical center. Put $Z = C_8 + C_8'$. Then we have $\mathrm{mult}_Z(\lambda\mathscr{M}_X) > 1$. Let $\pi : \widetilde{X} \to X$ be the blowup of X along Z, E the exceptional divisor, and H the pullback of a general hyperplane section on X to $\widetilde{X}$. Similarly as before, the assumption that C_8 and C_8' are non-canonical centers implies that $(3H-E)^2$ is effective. Using equations, we check that the linear system $|\mathscr{O}_X(5) - C_8 - C_8'|$ does not have base curves other than $C_8 + C_8'$. It follows that $(5H-E)$ is nef. Let us compute the intersection number $(3H-E)^2 \cdot (5H-E)$. We have

- $H^3 = 2$,
- $H^2 \cdot E = 0$,
- $H \cdot E^2 = -\deg(C_8 + C_8') = -16$,
- $E^3 = -\deg(\mathscr{N}_{C_8/X}) - \deg(\mathscr{N}_{C_8'/X}) = K_X \cdot (C_8 + C_8') + 4 = -44$.

Then

$$\begin{aligned}(3H-E)^2 \cdot (5H-E) &= 45H^3 - 39H^2E + 11HE^2 - E^3 \\ &= 90 - 0 - 176 + 44 \\ &= -42.\end{aligned}$$

This contradicts the fact that $(5H-E)$ is nef. □

On the other hand, C_4+C_4' can indeed be a non-canonical center. We present the Sarkisov link centered at C_4+C_4'. One can check that the linear system $|\mathscr{O}_X(3)-(C_4+C_4')|$ gives rise to a rational map $\tau : X \dashrightarrow \mathbb{P}^3$ fitting into a G'-equivariant commutative diagram

$$\begin{array}{ccc} & \widetilde{X} & \\ {}^{\pi}\swarrow & & \searrow^{\rho} \\ X & \overset{\tau}{\dashrightarrow} & \mathbb{P}^3 \end{array}$$

where π is the blowup along C_4+C_4' and the resolution ρ of τ is a double cover ramified along a singular sextic surface. The involution of this double cover gives rise to a G'-equivariantly birational involution

$$\delta : X \dashrightarrow X.$$

Note that δ naturally commutes with G' in $\mathrm{Bir}^{G'}(X)$. It follows that $\delta \notin \mathrm{Aut}(X) = \mathrm{PSO}_5(\mathbb{C})$ since no element in $\mathrm{Aut}(X)$ centralizes G'. Similarly as in Lemma 4.5.13, we can show that $-K_{\widetilde{X}}$ is big and nef, and that $|n(-K_{\widetilde{X}})|$ gives a small birational map for $n \gg 0$. Namely, δ also fits into a Sarkisov link as in (17).

Proof of Theorem 4.1.3. Note that the Sarkisov link centered at C_4+C_4' is again a G'-equivariantly birational selfmap. Thus, Proposition 4.7.2 and Theorem 4.7.1 still hold for the G'-action. The argument in the previous subsection applies and Theorem 4.1.3 is proved in the same way. □

Appendix: Magma code

We provide the Magma code that computes invariant curves for the non-standard action of $\mathfrak{A}_5$ on smooth quadric threefolds.

```
K:=CyclotomicField(30);

//e30:=259;
//C41,C42,C81,C82,C161,C162
G:=MatrixGroup<5, K | [
    Matrix(SparseMatrix(K, 5, 5, [
    <1, 2, 1>, <1, 5, -1>, <2, 4, 1>, <2, 5, -1>, <3, 5, -1>, <4,
        1, 1>, <4, 5,
    -1>, <5, 3, 1>, <5, 5, -1>])),
    Matrix(SparseMatrix(K, 5, 5, [
```

```
    <1, 2, -1>, <1, 3, 1>, <2, 2, -1>, <2, 5, 1>, <3, 2, -1>, <3, 4, 1>, <4, 1,
    1>, <4, 2, -1>, <5, 2, -1>]))
]> where w := K.1 where K := CyclotomicField(30);

P4<x1,x2,x3,x4,x5>:=ProjectiveSpace(K,4);
e30:=RootOfUnity(30);
c3:=[x1^2*x2 + x1*x2^2 + 2*x1*x2*x3 + x2^2*x3 + x2*x3^2 + 2*x2*x3*x4 +
            x3^2*x4 + x3*x4^2 + x1^2*x5 + 2*x1*x2*x5 + 2*x1*x4*x5 + 2*x3*x4*x5 +
            x4^2*x5 + x1*x5^2 + x4*x5^2,
        x1^2*x3 + x1*x3^2 + x1^2*x4 + 2*x1*x2*x4 + x2^2*x4 + 2*x1*x3*x4 +
            x1*x4^2 + x2*x4^2 + x2^2*x5 + 2*x1*x3*x5 + 2*x2*x3*x5 + x3^2*x5 +
            2*x2*x4*x5 + x2*x5^2 + x3*x5^2];
e5:=RootOfUnity(5);
X:=Scheme(P4, x1^2 + x1*x2 + x2^2 + x1*x3 + x2*x3 + x3^2 + x1*x4 + x2*x4 + x3*x4 + x4^2 +
        x1*x5 + x2*x5 + x3*x5 + x4*x5 + x5^2);
a3:=-e30^7 + e30^3 + e30^2 + 1;
a4:=e30^7 - e30^3 - e30^2 + 2;

f1:=a3*c3[1]+c3[2];
f2:=a4*c3[1]+c3[2];
X:=Q;
X1:=Scheme(P4,f1);
X2:=Scheme(P4,f2);
C41:=SingularSubscheme(X1);
df1:=DefiningEquations(C41);
C42:=SingularSubscheme(X2);

M:=GL(5,K);


inv1:=map<P4->P4|
[
    1/3*(e30^7 - e30^3 - e30^2 - 2)*x1^2 + 1/3*(-2*e30^7 +
        2*e30^3 + 2*e30^2 + 1)*x1*x2 + 1/3*(-2*e30^7 + 2*e30^3 +
        2*e30^2 + 1)*x2^2 + 1/3*(e30^7 - e30^3 - e30^2 +
        1)*x1*x3 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x2*x3 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x3^2 + 1/3*(e30^7 -
        e30^3 - e30^2 + 1)*x1*x4 + 1/3*(e30^7 - e30^3 -
```

```
        e30^2 + 1)*x2*x4 + 1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 +
        1)*x3*x4 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x4^2 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x1*x5 +
        1/3*(-5*e30^7 + 5*e30^3 + 5*e30^2 + 1)*x2*x5 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x3*x5 + 1/3*(-2*e30^7 +
        2*e30^3 + 2*e30^2 + 1)*x4*x5 + 1/3*(-2*e30^7 + 2*e30^3 +
        2*e30^2 + 1)*x5^2,
    1/3*(e30^7 - e30^3 - e30^2 - 2)*x1^2 + 1/3*(4*e30^7 -
        4*e30^3 - 4*e30^2 - 5)*x1*x2 + 1/3*(e30^7 - e30^3 -
        e30^2 - 2)*x2^2 + 1/3*(e30^7 - e30^3 - e30^2 - 5)*x1*x3
        + 1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 - 5)*x2*x3 +
        1/3*(e30^7 - e30^3 - e30^2 - 2)*x3^2 + 1/3*(e30^7 -
        e30^3 - e30^2 - 5)*x1*x4 + 1/3*(e30^7 - e30^3 -
        e30^2 - 5)*x2*x4 + 1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 -
        5)*x3*x4 + 1/3*(e30^7 - e30^3 - e30^2 - 2)*x4^2 +
        1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 - 5)*x1*x5 + 1/3*(e30^7
        - e30^3 - e30^2 - 5)*x2*x5 + 1/3*(e30^7 - e30^3 -
        e30^2 - 5)*x3*x5 + 1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 -
        5)*x4*x5 + 1/3*(e30^7 - e30^3 - e30^2 - 2)*x5^2,
    1/3*(e30^7 - e30^3 - e30^2 + 1)*x1^2 + 1/3*(-2*e30^7 +
        2*e30^3 + 2*e30^2 + 1)*x1*x2 + 1/3*(-2*e30^7 + 2*e30^3 +
        2*e30^2 + 1)*x2^2 + 1/3*(e30^7 - e30^3 - e30^2 +
        1)*x1*x3 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x2*x3 +
        1/3*(e30^7 - e30^3 - e30^2 - 2)*x3^2 + 1/3*(e30^7 -
        e30^3 - e30^2 + 1)*x1*x4 + 1/3*(-5*e30^7 + 5*e30^3 +
        5*e30^2 + 1)*x2*x4 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x3*x4 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x4^2 +
        1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 + 1)*x1*x5 + 1/3*(e30^7
        - e30^3 - e30^2 + 1)*x2*x5 + 1/3*(e30^7 - e30^3 -
        e30^2 + 1)*x3*x5 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x4*x5 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x5^2,
    1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x1^2 + 1/3*(-2*e30^7
        + 2*e30^3 + 2*e30^2 + 1)*x1*x2 + 1/3*(e30^7 - e30^3 -
        e30^2 + 1)*x2^2 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x1*x3
        + 1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 + 1)*x2*x3 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x3^2 + 1/3*(-5*e30^7 +
        5*e30^3 + 5*e30^2 + 1)*x1*x4 + 1/3*(e30^7 - e30^3 -
        e30^2 + 1)*x2*x4 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x3*x4 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x4^2 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x1*x5 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x2*x5 + 1/3*(e30^7 -
        e30^3 - e30^2 + 1)*x3*x5 + 1/3*(-2*e30^7 + 2*e30^3 +
        2*e30^2 + 1)*x4*x5 + 1/3*(e30^7 - e30^3 - e30^2 -
        2)*x5^2,
```

```
    1/3*(e30^7 - e30^3 - e30^2 + 1)*x1^2 + 1/3*(4*e30^7 -
        4*e30^3 - 4*e30^2 + 1)*x1*x2 + 1/3*(e30^7 - e30^3 -
        e30^2 + 1)*x2^2 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x1*x3
        + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x2*x3 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x3^2 + 1/3*(e30^7
        - e30^3 - e30^2 + 1)*x1*x4 + 1/3*(e30^7 - e30^3 -
        e30^2 + 1)*x2*x4 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x3*x4 + 1/3*(e30^7 - e30^3 - e30^2 - 2)*x4^2 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x1*x5 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x2*x5 + 1/3*(-5*e30^7 +
        5*e30^3 + 5*e30^2 + 1)*x3*x5 + 1/3*(-2*e30^7 + 2*e30^3 +
        2*e30^2 + 1)*x4*x5 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x5^2
]>;



inv2:=map<P4->P4|[
    1/3*(e30^7 - e30^3 - e30^2 - 2)*x1^2 + 1/3*(e30^7 -
        e30^3 - e30^2 + 1)*x1*x2 + 1/3*(e30^7 - e30^3 -
        e30^2 + 1)*x2^2 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x1*x3 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x2*x3 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x3^2 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x1*x4 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x2*x4 +
        1/3*(-5*e30^7 + 5*e30^3 + 5*e30^2 + 1)*x3*x4 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x4^2 + 1/3*(e30^7
        - e30^3 - e30^2 + 1)*x1*x5 + 1/3*(4*e30^7 - 4*e30^3 -
        4*e30^2 + 1)*x2*x5 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x3*x5 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x4*x5 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x5^2,
    1/3*(e30^7 - e30^3 - e30^2 - 2)*x1^2 + 1/3*(e30^7 -
        e30^3 - e30^2 - 5)*x1*x2 + 1/3*(e30^7 - e30^3 -
        e30^2 - 2)*x2^2 + 1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 -
        5)*x1*x3 + 1/3*(e30^7 - e30^3 - e30^2 - 5)*x2*x3 +
        1/3*(e30^7 - e30^3 - e30^2 - 2)*x3^2 + 1/3*(4*e30^7 -
        4*e30^3 - 4*e30^2 - 5)*x1*x4 + 1/3*(4*e30^7 - 4*e30^3 -
        4*e30^2 - 5)*x2*x4 + 1/3*(e30^7 - e30^3 - e30^2 -
        5)*x3*x4 + 1/3*(e30^7 - e30^3 - e30^2 - 2)*x4^2 +
        1/3*(e30^7 - e30^3 - e30^2 - 5)*x1*x5 + 1/3*(4*e30^7 -
        4*e30^3 - 4*e30^2 - 5)*x2*x5 + 1/3*(4*e30^7 - 4*e30^3 -
        4*e30^2 - 5)*x3*x5 + 1/3*(e30^7 - e30^3 - e30^2 -
        5)*x4*x5 + 1/3*(e30^7 - e30^3 - e30^2 - 2)*x5^2,
    1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x1^2 + 1/3*(e30^7 -
```

```
        e30^3 - e30^2 + 1)*x1*x2 + 1/3*(e30^7 - e30^3 -
        e30^2 + 1)*x2^2 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x1*x3 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x2*x3 +
        1/3*(e30^7 - e30^3 - e30^2 - 2)*x3^2 + 1/3*(-2*e30^7 +
        2*e30^3 + 2*e30^2 + 1)*x1*x4 + 1/3*(4*e30^7 - 4*e30^3 -
        4*e30^2 + 1)*x2*x4 + 1/3*(e30^7 - e30^3 - e30^2 +
        1)*x3*x4 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x4^2 +
        1/3*(-5*e30^7 + 5*e30^3 + 5*e30^2 + 1)*x1*x5 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x2*x5 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x3*x5 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x4*x5 + 1/3*(-2*e30^7 +
        2*e30^3 + 2*e30^2 + 1)*x5^2,
    1/3*(e30^7 - e30^3 - e30^2 + 1)*x1^2 + 1/3*(e30^7 -
        e30^3 - e30^2 + 1)*x1*x2 + 1/3*(-2*e30^7 + 2*e30^3 +
        2*e30^2 + 1)*x2^2 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 +
        1)*x1*x3 + 1/3*(-5*e30^7 + 5*e30^3 + 5*e30^2 + 1)*x2*x3 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x3^2 +
        1/3*(4*e30^7 - 4*e30^3 - 4*e30^2 + 1)*x1*x4 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x2*x4 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x3*x4 + 1/3*(e30^7 -
        e30^3 - e30^2 + 1)*x4^2 + 1/3*(e30^7 - e30^3 - e30^2
        + 1)*x1*x5 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x2*x5 +
        1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x3*x5 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x4*x5 + 1/3*(e30^7 -
        e30^3 - e30^2 - 2)*x5^2,
    1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2 + 1)*x1^2 + 1/3*(-5*e30^7
        + 5*e30^3 + 5*e30^2 + 1)*x1*x2 + 1/3*(-2*e30^7 + 2*e30^3
        + 2*e30^2 + 1)*x2^2 + 1/3*(-2*e30^7 + 2*e30^3 + 2*e30^2
        + 1)*x1*x3 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x2*x3 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x3^2 + 1/3*(-2*e30^7 +
        2*e30^3 + 2*e30^2 + 1)*x1*x4 + 1/3*(-2*e30^7 + 2*e30^3 +
        2*e30^2 + 1)*x2*x4 + 1/3*(e30^7 - e30^3 - e30^2 +
        1)*x3*x4 + 1/3*(e30^7 - e30^3 - e30^2 - 2)*x4^2 +
        1/3*(e30^7 - e30^3 - e30^2 + 1)*x1*x5 + 1/3*(-2*e30^7 +
        2*e30^3 + 2*e30^2 + 1)*x2*x5 + 1/3*(4*e30^7 - 4*e30^3 -
        4*e30^2 + 1)*x3*x5 + 1/3*(e30^7 - e30^3 - e30^2 +
        1)*x4*x5 + 1/3*(e30^7 - e30^3 - e30^2 + 1)*x5^2
]>;

C81:=inv1(C42);
[Degree(C81)];
C82:=inv2(C41);
[Degree(C82)];
```

```
C161:=inv2(C81);
[Degree(C161)];

C162:=inv1(C82);
[Degree(C162)];

extrainv:=Matrix(K
    ,5,5,[0,0,1,0,-1,0,0,0,1,-1,1,0,0,0,-1,0,1,0,0,-1,0,0,0,0,-1])
    ;
nG:=sub<M|G,extrainv>;
```

Chapter 5

Finite abelian groups acting on rationally connected threefolds: groups of K3 type

"I maths cool bas lezm neshrab may."

Line Kerbage

We study finite abelian groups acting on three-dimensional rationally connected varieties. We concentrate on groups of K3 type, that is, abelian extensions of groups that faithfully act on a K3 surface by a cyclic group. In particular, if a finite abelian group faithfully acts on a $G\mathbb{Q}$-Fano threefold while preserving a K3 surface (with at worst du Val singularities), then it is of K3 type. We prove a classification theorem for the groups of K3 type which can act on three-dimensional rationally connected varieties. The results presented in this chapter have been obtained in collaboration with Konstantin Loginov and Zhijia Zhang, see Loginov et al. (2025). All authors have approved the inclusion of this work in the present thesis and acknowledge equal contribution.

5.1 Introduction

By $\mathrm{Bir}(X)$ we denote the group of birational automorphisms of an algebraic variety X. We deal with the classification problem of finite subgroups in $\mathrm{Bir}(X)$ when X is a rationally connected variety. More specifically, we are most interested in finite subgroups of the Cremona group. Recall that the Cremona group is defined as $\mathrm{Cr}_n(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^n)$. The classification of finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ was obtained in Dolgachev and Iskovskikh (2009). As for finite subgroups of $\mathrm{Cr}_3(\mathbb{C})$, the complete classification seems to be out of reach. There exist results concerning some classes of finite groups, see Y. Prokhorov (2012) in the case of simple groups.

In this paper, we concentrate on finite abelian subgroups of $\mathrm{Bir}(X)$ when X is a rationally connected variety. The case of $\mathrm{Cr}_1(\mathbb{C}) = \mathrm{PGL}(2,\mathbb{C})$ is elementary, see Proposition 6. The classification of finite abelian subgroups in $\mathrm{Cr}_2(\mathbb{C})$ was obtained in Blanc (2007), see Theorem 10. The study of finite abelian subgroups in the case of $\mathrm{Cr}_3(\mathbb{C})$ was initiated in Loginov (2024). In arbitrary dimension, there exists the following result.

Theorem 1 ((Kollár & Zhuang, 2024, Corollary 11)). Let X be a rationally connected variety of dimension n, and let $G \subset \mathrm{Bir}(X)$ be a finite abelian p-group. Then G can be generated by r elements where

$$r \leq \frac{pn}{p-1} \leq 2n.$$

The bound in Theorem 1 in dimension $n = 2$ was obtained in Beauville (2007), it also follows from Blanc (2007) and Dolgachev and Iskovskikh (2009); in dimension $n = 3$ it was obtained in a series of works, see Y. Prokhorov (2011), Y. Prokhorov (2014), Y. Prokhorov and Shramov (2018), Kuznetsova (2020), Xu (2020), Loginov (2022). They use explicit methods of the minimal model program.

To deal with the case $n = 3$, the following definitions were introduced in Loginov (2024). A finite abelian group G is called *a group of product type*, if $G = G_1 \times G_2$ where $G_i \subset \mathrm{Cr}_i(\mathbb{C})$. In particular, G is isomorphic to a subgroup in

$$\mathrm{Cr}_1(\mathbb{C}) \times \mathrm{Cr}_2(\mathbb{C}) \subset \mathrm{Cr}_3(\mathbb{C}).$$

Using the classification of finite abelian subgroups of $\mathrm{Cr}_i(\mathbb{C})$ for $i = 1, 2$, see Proposition 6 and Theorem 10, it is not hard to write down the complete list of groups of product type, see Table 1 in Section 5.2.3.

We say that a finite abelian group G is of *K3 type*, if G is an abelian extension of a finite abelian group H that faithfully acts on a K3 surface, by a cyclic group:

$$0 \to \mathbb{Z}_m \to G \to H \to 0. \quad (2)$$

In particular, if a finite abelian group faithfully acts on a three-dimensional variety while preserving a K3 surface (with at worst du Val singularities), then it is of K3 type.

Theorem 3 ((Loginov, 2022, Theorem 1.7)). Let X be a rationally connected variety of dimension 3, and let $G \subset \mathrm{Bir}(X)$ be a finite abelian group. Then

1. either G is of product type,
2. or G is of K3 type,
3. or G faithfully acts on a $G\mathbb{Q}$-Fano threefold X' with $|-K_{X'}| = \emptyset$ such that X' is G-birational to X. Moreover, any $G\mathbb{Q}$-Mori fiber space with a faithful action of G is a $G\mathbb{Q}$-Fano threefold with empty anti-canonical system.

Three cases in Theorem 3 are not mutually exclusive. It is known that if a finite abelian group that faithfully acts on a rationally connected threefold preserving a rational curve, a rational surface, or a structure of a Mori fiber space with a non-trivial base, then such a group is of product type, see (Loginov, 2024, Corollary 3.14, Corollary 3.17). Essentially, this follows from a purely algebraic result on abelian extensions of finite abelian groups, see Proposition 7. It is also known that if a finite abelian group that faithfully acts on a threefold with terminal singularities has a (smooth or singular) fixed point then it is of product type, see Theorem 32. However, not all finite abelian groups that can faithfully act on a rationally connected threefold are of product type.

In (Loginov, 2022, Corollary 1.10) it is proven that there are only finitely many isomorphism classes of finite abelian groups of K3 type which faithfully act on a rationally connected threefold. This result follows from two boundedness results. First, there are only finitely many isomorphism classes of finite groups that can faithfully act on a K3 surface, see Brandhorst and Hofmann (2023) for the complete classification. This bounds H in the exact sequence in (2). The second result, which is needed to bound m in (2), is the boundedness of the indices of Fano threefolds with canonical singularities. Of course, this follows from the boundedness of Fano threefolds with canonical singularities. However, it is effectively known only in the case of isolated canonical singularities, in which case it is equal to 61, see Jiang and Liu (2025). In the case of non-isolated canonical singularities, there exists a bound 228614400, see (Birkar, 2019, Lemma 2.3). It seems far from being sharp. Hence the main problem in dealing with groups of K3 type that act on rationally connected threefolds is to bound the number m in (2) for each group H that can faithfully act on a K3 surface. It turns out that this number indeed can be effectively bounded.

The main result of our work is as follows.

Theorem 4. Let X be a rationally connected variety of dimension 3, and let $G \subset \mathrm{Bir}(X)$ be a finite abelian group. Then either G is of product type, or of type (3) as in Theorem 3, or G is isomorphic to one of the following groups:

1. $\mathbb{Z}_4^4$,

2. $\mathbb{Z}_6^3 \times \mathbb{Z}_2$,
3. $\mathbb{Z}_6^2 \times \mathbb{Z}_3^2$,
4. $\mathbb{Z}_8^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$.

All the cases in Theorem 4 are realized as shown in Example 3. It is expected that there are no groups of the third type in Theorem 3 which are not of product type or of K3 type, see (Loginov, 2024, Conjecture 1.8). If this conjecture is true, then the list of group of product type ((Loginov, 2024, Table 1)) together with Theorem 4 provide the complete list of finite abelian groups that can act on a rationally connected variety of dimension 3.

Corollary 1. Let X be a rationally connected variety of dimension 3, and let $G \subset \mathrm{Bir}(X)$ be a finite abelian group. Assume that

- either (Loginov, 2024, Conjecture 1.8) holds,
- or G acts faithfully on a $G\mathbb{Q}$-Fano threefold X' with $|-K_{X'}| \neq 0$.

Then G is isomorphic to one of the following groups (and all these cases are realized):

	G	
(1)	$\mathbb{Z}_k \times \mathbb{Z}_l \times \mathbb{Z}_m$	$k \geq 1,\ l \geq 1,\ m \geq 1$
(2)	$\mathbb{Z}_2k \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$k \geq 1$
(3)	$\mathbb{Z}_3k \times \mathbb{Z}_3^3$	$k \geq 1$
(4)	$\mathbb{Z}_2k \times \mathbb{Z}_2l \times \mathbb{Z}_2^2$	$k \geq 1,\ l \geq 1$
(5)	$\mathbb{Z}_2n \times \mathbb{Z}_2^4$	$n \geq 1$
(6)	$\mathbb{Z}_4^2 \times \mathbb{Z}_2^3$	
(7)	$\mathbb{Z}_2^6$	
(8)	$\mathbb{Z}_4^4$	
(9)	$\mathbb{Z}_6^3 \times \mathbb{Z}_2$	
(10)	$\mathbb{Z}_6^2 \times \mathbb{Z}_3^2$	
(11)	$\mathbb{Z}_8^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	

Table 1. Conjectural list of all finite abelian groups that can act on a rationally connected threefold

In Table 1, the groups (1)–(7) are of product type, so they can act on a rational threefold, while the groups (8)–(11) are of K3 type and not of product type.

Finite abelian groups of symplectic automorphisms of K3 surfaces were classified by V. Nikulin in the famous paper Nikulin (1980a), see Theorem 18. The classification of Brandhorst and Hofmann (2023) provides the list of all maximal finite abelian groups that can faithfully act on a K3 surface, cf. Theorem 19. It turns out that all but 6 of them can be realized as subgroups of $\mathrm{Cr}_2(\mathbb{C})$.

Proposition 2. Let H be a finite abelian group that faithfully acts on a K3 surface. Assume further that H is not isomorphic to a subgroup of $\mathrm{Cr}_2(\mathbb{C})$, that is, it cannot faithfully act on a rational surface. Then H is isomorphic to one of the following groups:

1. $\mathbb{Z}_4^3$,
2. $\mathbb{Z}_6^2 \times \mathbb{Z}_2$,
3. $\mathbb{Z}_6 \times \mathbb{Z}_3^2$,
4. $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$,
5. $\mathbb{Z}_2^5$,
6. $\mathbb{Z}_4 \times \mathbb{Z}_2^3$.

Using the results of Brandhorst and Hofmann (2023), we give a more precise description of the action of these 6 groups on K3 surfaces, including the decomposition into symplectic and non-symplectic subgroups and invariant lattice in cohomology of a surface, see Corollary 13 and Proposition 20. In particular, all of these 6 groups are not symplectic. In fact, all the finite abelian groups of symplectic automorphisms can be realized as subgroups of $\mathrm{Cr}_2(\mathbb{C})$.

Using the exact sequence (2) and Proposition 7, we conclude that if H is not one of 6 groups from Proposition 2 then G is of product type. Hence, to study groups of K3 type which are not or product type, we may assume that H is one of the 6 groups as in Proposition 2. The problem is to bound the number m as in (2). As shown in (Loginov, 2022, Proposition 11.3), this number can be bounded by the maximal index of Fano threefolds with canonical singularities.

Example 3. We construct the actions on (singular) Fano varieties in weighted projective spaces of groups of K3 type.

1. Let $X_4 \subset \mathbb{P}^4$ be given by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0,$$

with the action of $G = \mathbb{Z}_4^4$. Note that X_4 is smooth.

2. Let $X_6 \subset \mathbb{P}(1,1,1,1,3)$ be given by the equation

$$x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^2 = 0,$$

with the action of $G = \mathbb{Z}_6^3 \times \mathbb{Z}_2$. Note that X_6 is smooth.

3. Let $X_6' \subset \mathbb{P}(1,1,1,2,2)$ be given by the equation

$$x_0^6 + x_1^6 + x_2^6 + x_3^3 + x_4^3 = 0,$$

with the action of $G = \mathbb{Z}_6^2 \times \mathbb{Z}_3^2$. Note that X_6' has 3 singular points of type $1/2 \times (1,1,1)$.

4. Let $X_8 \subset \mathbb{P}(1,1,1,2,4)$ be given by the equation

$$x_0^8 + x_1^8 + x_2^8 + x_3^4 + x_4^2 = 0,$$

with the action of $G = \mathbb{Z}_8^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$. Note that X_8 has 2 singular points of type $1/2 \times (1,1,1)$.

5. Let $X_{2,2,2} \subset \mathbb{P}^6$ be the intersection of three quadrics, so it is given by the equations

$$\sum_{i=0}^{6} x_i^2 = \sum_{i=0}^{6} \lambda_i x_i^2 = \sum_{i=0}^{6} \mu_i x_i^2 = 0,$$

with the action of $G = \mathbb{Z}_2^6$.

6. Let $X_{4,4} \subset \mathbb{P}(1,1,1,2,2,2)$ be given by the equations

$$x_0^4 + x_1^4 + x_2^4 + x_3^2 + x_4^2 + x_5^2 = \lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 x_5^2 = 0,$$

with the action of $G = \mathbb{Z}_4^2 \times \mathbb{Z}_2^3$. Note that $X_{4,4}$ has 4 singular points of type $1/2 \times (1,1,1)$.

By (Cheltsov & Park, 2017, Corollary 1.1.9) the varieties X in cases (1)–(4)[1] in Example 3 are not rational. This observation gives rise to the following question.

Question 4. Can the groups (1)–(4) as in Example 3 be realized as subgroups of $\mathrm{Cr}_3(\mathbb{C})$? In other words, can such groups faithfully act on a rational threefold?

We note that the Fermat quartic K3 surface S_4 in $\mathbb{P}^3$ which is a G-invariant hyperplane section of X_4 from Example 3.(1) enjoys many nice properties. For example, it has maximal possible Picard rank 20, see Example 18. Recall that such K3 surfaces are called *singular* (although later in the text we reserve this term for surfaces having singularities to avoid confusion). In fact S_4 is a Kummer K3 surface associated with the product of two isogenous elliptic curves E_i and E_{2i}. In Loginov (2024) it is shown that the "exceptional" finite abelian groups in $\mathrm{Cr}_2(\mathbb{C})$, that is group (3)–(5) from Theorem 10, correspond to elliptic curves with complex multiplication. As pointed out by Schütt (2008), singular K3 surfaces in many ways behave like elliptic curves with complex multiplication. Let $S_6, S_6', S_8, S_{2,2,2}, S_{4,4}$ be the G-invariant hyperplane sections given by the equation $x_0 = 0$ of Fano threefolds (2)–(6) from Example 3. Using the results of Esser and Li (2025), one easily computes $\rho(S_6) = \rho(S_6') = 20$, $\rho(S_8) = 18$. This observation motivates the following question.

Question 5. Are $S_6, S_6', S_8, S_{2,2,2}, S_{4,4}$ Kummer K3 surfaces?

It is known that singular K3 surfaces are classified by its transcendental lattice T_S which is even, positive definite and has rank 2. If the values of the quadratic form on this lattice are divisible by 4, then S is Kummer. Similar criteria are known for K3 surfaces with Picard rank at least 17, cf. (Huybrechts, 2016, 14.3.20). It would be interesting to compute these lattices for the above surfaces.

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1. KL: (6) as well. What about (5)?

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5.2 Preliminaries

We work over the field of complex numbers $\mathbb{C}$. All the varieties are projective and defined over $\mathbb{C}$ unless stated otherwise. We will use the language of the minimal model program (the MMP for short), see e.g. Kollár and Mori (1998).

5.2.1 Group actions

We start with the following well known results.

Lemma 5 (cf. (Popov, 2014, Lemma 4)). Let X be an algebraic variety, and G be a finite group such that $G \subset \mathrm{Aut}(X)$. Assume $P \in X$ is a fixed point of G. Then the induced action of G on the tangent space $T_P X$ is faithful.

By $\mathfrak{r}(G)$ we denote the *rank* of a group G, that is, the minimal number of generators.

Lemma 6 (cf. (Y. Prokhorov, 2011, Lemma 2.6),(Loginov, 2024, Lemma 2.8)). Let X be a three-dimensional algebraic variety with isolated singularities, and G be a finite abelian group such that $G \subset \mathrm{Aut}(X)$.

1. If there is a curve $C \subset X$ of G-fixed points, then $\mathfrak{r}(G) \leq 2$.
2. If there is a (possibly, reducible) divisor $S \subset X$ of G-fixed points, then $\mathfrak{r}(G) \leq 1$. If moreover S is singular along a curve, then G is trivial.
3. If X is smooth, and $S \subset X$ is a divisor of G-fixed points such that S is singular, then G is trivial.

5.2.2 Extensions of finite abelian groups

Let G be a finite abelian group. In what follows, we will denote by G_p the p-Sylow subgroup of G where p is a prime number, so we have

$$G = \prod_{p \geq 2} G_p.$$

We occasionally say that G_p is the p*-part* of G. For an abelian p-group G_p, we say that G_p has type

$$\lambda = [\lambda_1, \ldots, \lambda_k] \qquad \text{for} \qquad \lambda_1 \geq \ldots \geq \lambda_k \geq 0$$

if

$$G_p = \mathbb{Z}_p^{\lambda_1} \times \ldots \times \mathbb{Z}_p^{\lambda_k}.$$

Note that the type of an abelian p-group is defined uniquely. Also, a sequence of finite abelian groups

$$0 \to H \to G \to K \to 0 \tag{7}$$

is exact if and only if for any prime p the p-parts H_p, G_p, K_p of the groups H, G, K, respectively, form an exact sequence

$$0 \longrightarrow H_p \longrightarrow G_p \longrightarrow K_p \longrightarrow 0 \tag{8}$$

We say that the exact sequence (8) is the *p-part* of the exact sequence (7).

To any type $\lambda = [\lambda_1, \ldots, \lambda_k]$ corresponds the Young diagram with λ_i squares in the i-th row. For two Young diagrams $\lambda = [\lambda_1, \ldots, \lambda_k]$ and $\mu = [\mu_1, \ldots \mu_k]$, one can define their product $\lambda \cdot \mu$ as a formal linear combination of Young diagrams with non-negative coefficients, see e.g. (Fulton, 2000, Section 2). Then the *Littlewood–Richardson coefficient* $c^{\nu}_{\lambda\mu}$ is the coefficient at the Young diagram $\nu = [\nu_1, \ldots \nu_k]$ in the product of Young diagrams $\lambda \cdot \mu$.

We recall the following criterion, which gives all the possible isomorphism classes for G to fit into an exact sequence as above, in the case of finite abelian p-groups.

Theorem 9 ((Fulton, 2000, Section 2)). Let G_p, H_p, and K_p be finite abelian p-groups, respectively of types $\mu = (\mu_1, \ldots, \mu_k), \lambda = (\lambda_1, \ldots, \lambda_k)$, and $\nu = (\nu_1, \ldots, \nu_k)$. Then an extension of the form

$$1 \to H_p \to G_p \to K_p \to 1$$

exists if and only if for the Littlewood-Richardson coefficient we have $c^{\mu}_{\lambda\nu} > 0$.

5.2.3 Groups or product type

We recall the results on finite abelian subgroups of Cremona groups in lower dimensions. The one-dimensional case is elementary.

Proposition 6. Let G be a finite abelian subgroup of $\mathrm{Cr}_1(\mathbb{C})$, which is isomorphic to $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2, \mathbb{C})$. Then G is isomorphic to one of the following groups:

1. $\mathbb{Z}_n, \ n \geq 1$,
2. $\mathbb{Z}_2^2$.

Theorem 10 (Blanc (2007)). Let G be a finite abelian subgroup of $\mathrm{Cr}_2(\mathbb{C})$. Then G is isomorphic to one of the following groups:

1. $\mathbb{Z}_n \times \mathbb{Z}_m, \ n \geq 1, \ m \geq 1$,
2. $\mathbb{Z}_2 n \times \mathbb{Z}_2^2, \ n \geq 1$,
3. $\mathbb{Z}_4^2 \times \mathbb{Z}_2$,
4. $\mathbb{Z}_3^3$,
5. $\mathbb{Z}_2^4$.

Definition 11. We say that a finite abelian group G is *a group of product type* if $G = G_1 \times G_2$ where $G_i \subset \mathrm{Cr}_i(\mathbb{C})$. In particular, G is isomorphic to a subgroup in

$$\mathrm{Cr}_1(\mathbb{C}) \times \mathrm{Cr}_2(\mathbb{C}) \subset \mathrm{Cr}_3(\mathbb{C}).$$

Using Proposition 6 and Theorem 10, we obtain

	G	
(1)	$\mathbb{Z}_k \times \mathbb{Z}_l \times \mathbb{Z}_m$	$k \geq 1,\ l \geq 1,\ m \geq 1$
(2)	$\mathbb{Z}_2k \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$k \geq 1$
(3)	$\mathbb{Z}_3k \times \mathbb{Z}_3^3$	$k \geq 1$
(4)	$\mathbb{Z}_2k \times \mathbb{Z}_2l \times \mathbb{Z}_2^2$	$k \geq 1,\ l \geq 1$
(5)	$\mathbb{Z}_2n \times \mathbb{Z}_2^4$	$n \geq 1$
(6)	$\mathbb{Z}_4^2 \times \mathbb{Z}_2^3$	
(7)	$\mathbb{Z}_2^6$	

Table 2. Groups of product type

Proposition 7 ((Loginov, 2024, Proposition 3.12))**.** Let $H \subset \mathrm{Cr}_1(\mathbb{K})$ and $K \subset \mathrm{Cr}_2(\mathbb{K})$ be finite abelian groups. Then an abelian extension G of H by K (or K by H) is of product type.

Definition 12. We say that a finite abelian group G is of *K3 type*, if G is an abelian extension of a finite abelian group H that faithfully acts on a K3 surface, by a cyclic group:

$$0 \to \mathbb{Z}_m \to G \to H \to 0.$$

5.2.4 Fermat complete intersections

Consider a weighted projective space

$$\mathbb{P} = \mathbb{P}(a_0^{r_0} : \ldots : a_M^{r_M})$$

where $a_i^{r_i}$ stands for $r_i \geq 1$ consecutive identical weights a_i, and $1 \leq a_0 \leq \ldots \leq a_M$. Put $N = \sum r_i a_i$. Then $\mathbb{P}$ is called well formed, if $\gcd(a_i) = 1$ for any set of $N-1$ numbers a_i. Let us recall the structure of the automorphism groups of weighted projective spaces, cf. Przyjalkowski and Shramov (2020).

Lemma 13. Let $\mathbb{P} = \mathbb{P}(a_0^{r_0}, \dots, a_M^{r_M})$ be a well formed weighted projective space, where $a_i^{r_i}$ stands for $r_i \geq 1$ consecutive identical weights a_i, and $1 \leq a_0 \leq \ldots \leq a_M$. Then $\mathrm{Aut}(P) = R \rtimes L$, where R is generated by automorphisms of the form

$$[x_{0,1} : \ldots : x_{0,r_0} : x_{1,1} : \ldots : x_{1,r_1} : \ldots : x_{p,1} : \ldots : x_{p,r_M}]$$
$$\downarrow$$
$$[x_{0,1} : \ldots : x_{0,r_0} : x_{1,1} + \varphi_{1,1} : \ldots : x_{1,r_1} + \varphi_{1,r_1} : \ldots : x_{p,1} + \varphi_{p,1} : \ldots : x_{p,r_M} + \varphi_{p,r_M}],$$

where each $\varphi_{p,q}$ is a polynomial in the variables $x_{i<p,j}$ of degree a_i on each variable $x_{i,j}$, and L is the quotient of $\mathrm{GL}_{r_0} \times \cdots \times \mathrm{GL}_{r_M}$ by $\{(t^{a_0} I_{r_0}, \dots, t^{a_M} I_{r_M}), t \in \mathrm{C}^\times\} = \mathrm{C}^\times$.

Definition 14. We define a *Fermat hypersurface* of degree d in a well formed weighted projective space $\mathbb{P} = \mathbb{P}(a_0^{r_0}, \dots, a_M^{r_M})$:

$$X_d = \left\{ \sum x_{i,j}^{d/a_i} = 0 \right\} \subset \mathbb{P},$$

where d is divisible by a_i for any i.

Similarly, a *Fermat complete intersection* of multidegree $d_1 \cdot \ldots \cdot d_k$ for $k \geq 1$ in a well formed weighted projective space $\mathbb{P}$ is given by

$$X_{d_1 \cdot \ldots \cdot d_k} = \left\{ \sum \lambda_{i,j;1} x_{i,j}^{d_1/a_i} = \ldots = \sum \lambda_{i,j;k} x_{i,j}^{d_k/a_i} = 0 \right\} \subset \mathbb{P},$$

where d_s is divisible by a_i for any s and any i, and $\lambda_{i,j;s} \in \mathbb{C}$.

Remark 8. Note that a Fermat hypersurface is a (singular) Fano variety if and only $d < \sum r_i a_i$. A Fermat complete intersection is a (singular) Fano variety if and only if $\sum d_i < \sum r_i a_i$.

Lemma 15. Let $X = X_d \subset \mathbb{P} = \mathbb{P}(a_0^{r_0} : \cdots : a_M^{r_M})$ be a Fermat hypersurface. Then the group

$$G = (\mathbb{Z}_{d/a_0}^{r_0} \times \ldots \times \mathbb{Z}_{d/a_M}^{r_M}) / \mathbb{Z}_d$$

faithfully acts on X.

Let $X' = X_{d_1 \ldots d_k} \subset \mathbb{P} = \mathbb{P}(a_0^{r_0} : \cdots : a_M^{r_M})$ be a Fermat complete intersection. Put $d' = \gcd(d_s)_{1 \leq s \leq k}$. Then the group

$$G = (\mathbb{Z}_{d'/a_0}^{r_0} \times \ldots \times \mathbb{Z}_{d'/a_M}^{r_0}) / \mathbb{Z}_d'$$

faithfully acts on X'.

Proof. Follows from Lemma 13. ☐

5.2.5 Lattices

We recall some generalities on lattices, see e.g. (Huybrechts, 2016, Chapter 14) or Nikulin (1980b). By a lattice Λ we mean a free finitely generated abelian group $\mathbb{Z}^n$ equipped with a symmetric bilinear form

$$B_\Lambda : \Lambda \times \Lambda \to \mathbb{Z}.$$

The lattice Λ is called even if $Q_\Lambda(v) := B_\Lambda(v,v)$ is even for any $v \in \Lambda$. The dual lattice Λ^* is defined as

$$\Lambda^* = \{v \in \Lambda \otimes \mathbb{Q} \,|\, B_\Lambda(v,w) \in \mathbb{Z} \text{ for any } w \in \Lambda\},$$

where the bilinear form B_Λ is naturally extended to $\Lambda \otimes \mathbb{Q}$. *The discriminant group* of Λ is defined as $A_\Lambda = \Lambda^*/\Lambda$. If B_Λ is non-degenerate then A_Λ is a finite abelian group. In this case, its order $\mathrm{disc}(\Lambda) = |A_\Lambda|$ is called *the discriminant* of Λ. Note that

$$\mathrm{disc}(\Lambda) = |\det B_\Lambda|,$$

where by abuse of notation we denote by B_Λ the Gram matrix of Λ.

In what follows, by $k\Lambda$ we will denote the lattice obtained from a lattice Λ by multiplying all its vectors by $k \in \mathbb{Z}$. In particular, we have $B_{k\Lambda} = k^2 B_\Lambda$.

5.2.6 Mori fiber space

Let G be a finite group. Recall that a normal projective G-variety X is called $G\mathbb{Q}$-factorial, if every G-invariant Weil divisor on X is $\mathbb{Q}$-Cartier. A $G\mathbb{Q}$-*Mori fiber space* is a $G\mathbb{Q}$-factorial variety X with at worst terminal singularities together with a G-equivariant contraction $f\colon X \to Z$ to a normal variety Z such that $\rho^G(X/Z) = 1$ and $-K_X$ is ample over Z. If Z is a point, we say that X is a $G\mathbb{Q}$-Fano variety.

5.3 Group actions on K3 surfaces

In this section, we consider actions of finite abelian groups on K3 surfaces. By a K3 surface we mean a normal projective surface S with at worst canonical singularities such that $\mathrm{H}^1(S, \mathscr{O}_S) = 0$ and $K_S \sim 0$.

Let $H \subset \mathrm{Aut}(S)$ be a finite group where S is a smooth projective K3 surface (we always can assume this by passing to the minimal resolution). There is a natural exact sequence (cf. Huybrechts (2016))

$$0 \to H_0 \to H \xrightarrow{\alpha} \mathbb{Z}_m \to 0, \tag{16}$$

where $\mathbb{Z}_m$ is a cyclic group that acts via multiplication by a primitive m-th root of unity on a non-zero holomorphic 2-form ω_S on S.

Definition 17. Let σ be a finite order automorphism of a K3 surface S. Then it is called a *symplectic automorphism*, if $\alpha(\sigma) = 1$. Otherwise, it is called *non-symplectic*. Moreover, we call σ *purely non-symplectic*, if $\mathrm{ord}(\alpha(\sigma)) = \mathrm{ord}(\sigma)$.

A group H acting on S is called *symplectic* (resp., *non-symplecic*, *purely non-symplectic*), if every non-trivial element of H is symplectic (resp., non-symplectic, purely non-symplectic).

Recall the following classical result.

Theorem 18 ((Nikulin, 1980a, 4.5)). In the exact sequence (16), the group H_0 is isomorphic to one of the following groups:

1. $\mathbb{Z}_n$, $1 \leq n \leq 8$,
2. $\mathbb{Z}_2 \times \mathbb{Z}_6$,
3. $\mathbb{Z}_3^2$,
4. $\mathbb{Z}_4^2$,
5. $\mathbb{Z}_2 \times \mathbb{Z}_4$,
6. $\mathbb{Z}_2^k$, $1 \leq k \leq 4$.

Proposition 9 (cf. (Huybrechts, 2016, 15.1.8))**.** A symplectic automorphism of finite order on a K3 surface has finitely many fixed points. More precisely, for such an automorphism σ, if we denote by $\mathrm{Fix}(\sigma)$ its fixed locus, we have

$\mathrm{ord}(\sigma)$	2	3	4	5	6	7	8
$\lvert\mathrm{Fix}(\sigma)\rvert$	8	6	4	4	2	3	2

Remark 10. A *Nikulin involution* is a symplectic automorphism of a smooth K3 surface of order 2. According to Proposition 9, the fixed locus of a Nikulin involution consists of exactly 8 points.

We denote by $\rho(S)$ the Picard rank of a (smooth) K3 surface S. It is well known that $1 \leq \rho(S) \leq 20$.

Proposition 11 (cf. (Huybrechts, 2016, 15.1.14))**.** In the exact sequence (16), the number m satisfies

$$\varphi(m) \leq \mathrm{rk}\, T_S = 22 - \rho(S),$$

and $\varphi(m) \,|\, \mathrm{rk}\, T_S$. In particular, $m \leq 66$.

Also, if a (purely) non-symplectic automorphism σ has order m, the list of all possibilities for m is given in (Brandhorst & Hofmann, 2023, Corollary 1.3). For example, if m is prime, then $m \in \{2, 3, 5, 7, 11, 13, 17, 19\}$.

Theorem 19 (Brandhorst and Hofmann (2023)). Let H be a finite abelian group that acts on a smooth K3 surface. Assume that H is not purely non-symplectic. If H is assumed to be maximal, then H is isomorphic to one of the following groups:

1. $\mathbb{Z}_4^3$,
2. $\mathbb{Z}_6^2 \times \mathbb{Z}_2$,
3. $\mathbb{Z}_6 \times \mathbb{Z}_3^2$,
4. $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$,
5. $\mathbb{Z}_2^5$,
6. $\mathbb{Z}_4 \times \mathbb{Z}_2^3$,
7. $\mathbb{Z}_12 \times \mathbb{Z}_6$,
8. $\mathbb{Z}_60$,
9. $\mathbb{Z}_10 \times \mathbb{Z}_5$,
10. $\mathbb{Z}_12 \times \mathbb{Z}_4$,
11. $\mathbb{Z}_12 \times \mathbb{Z}_2^2$,
12. $\mathbb{Z}_18 \times \mathbb{Z}_3$,
13. $\mathbb{Z}_15 \times \mathbb{Z}_3$,
14. $\mathbb{Z}_42$,
15. $\mathbb{Z}_30 \times \mathbb{Z}_2$,
16. $\mathbb{Z}_28 \times \mathbb{Z}_2$,
17. $\mathbb{Z}_24 \times \mathbb{Z}_2$,
18. $\mathbb{Z}_20 \times \mathbb{Z}_2$,
19. $\mathbb{Z}_18 \times \mathbb{Z}_2$,
20. $\mathbb{Z}_16 \times \mathbb{Z}_2$.

Remark 12. Among all the groups in Theorem 19, only the groups (1)–(6) are not isomorphic to a subgroup of $\mathrm{Cr}_2(\mathbb{C})$. However, all the groups in Theorem 19 are isomorphic to subgroups of $\mathrm{Cr}_3(\mathbb{C})$, and moreover, they are all of product type.

Corollary 13. Let H be the groups (1)–(6) from Proposition 19. Then the exact sequence (16) splits, so we have

$$H = H_0 \times \mathbb{Z}_m$$

where H_0 is the subgroup of symplectic automorphisms, and $\mathbb{Z}_m$ is the group whose non-trivial elements act purely non-symplectically. More precisely, one of the following holds

1. $H_0 = \mathbb{Z}_4^2, \quad m = 4,$
2. $H_0 = \mathbb{Z}_6 \times \mathbb{Z}_2, \quad m = 6,$
3. $H_0 = \mathbb{Z}_3^2, \quad m = 6,$
4. $H_0 = \mathbb{Z}_4 \times \mathbb{Z}_2, \quad m = 8,$
5. $H_0 = \mathbb{Z}_2^4, \quad m = 2,$
6. $H_0 = \mathbb{Z}_2^3, \quad m = 4.$

Proof. From the case by case analysis using Theorem 19, Theorem 18 and the exact sequence (16) we obtain the cases (1)–(6) and one exceptional case $H_0 = \mathbb{Z}_2^4, m = 2$. However, the latter case is not realized according to Brandhorst and Hofmann (2023). ☐

We collect some examples of actions of finite abelian groups on K3 surfaces.

Example 14. We start with the case of smooth K3 surfaces which is a Fermat complete intersections on which the following group G acts faithfully (cf. Lemma 15).

	K3 surface	**Group**
(1)	$X_6 \subset \mathbb{P}(1,1,1,3)$	$\mathbb{Z}_6^2 \times \mathbb{Z}_2$
(2)	$X_4 \subset \mathbb{P}^3$	$\mathbb{Z}_4^3$
(3)	$X_{2,2,2} \subset \mathbb{P}^5$	$\mathbb{Z}_2^5$

Table 3. Examples of smooth K3 surfaces with the action of a finite abelian group

Example 15. Now we treat the case of singular K3 surfaces, cf. Iano-Fletcher (2000). Taking the minimal resolution, we obtain a smooth K3 surface with the action of the group H. We also describe singularities of S. For example, we write $\mathrm{Sing}(S) = 3A_1 + 4A_2$, if X has 3 du Val singular points of type A_1 and 4 singular points of type A_2. To compute the group H one can use Lemma 15.

	K3 surface	Group	Singularities
(1)	$X_{4,4} \subset \mathbb{P}(1,1,2,2,2)$	$\mathbb{Z}_4 \times \mathbb{Z}_2^3$	$4A_1$
(2)	$X_8 \subset \mathbb{P}(1,1,2,4)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$2A_1$
(3)	$X_6 \subset \mathbb{P}(1,1,2,2)$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$3A_1$
(4)	$X_{12} \subset \mathbb{P}(1,3,4,4)$	$\mathbb{Z}_4 \times \mathbb{Z}_3^2$	$3A_3$
(5)	$X_{12} \subset \mathbb{P}(1,2,3,6)$	$\mathbb{Z}_6 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$2A_1 + 2A_2$
(6)	$X_{12} \subset \mathbb{P}(2,3,3,4)$	$\mathbb{Z}_4^2 \times \mathbb{Z}_3$	$3A_1 + 4A_2$
(7)	$X_{6,6} \subset \mathbb{P}(1,2,3,3,3)$	$\mathbb{Z}_3 \times \mathbb{Z}_2^3$	$4A_2$
(8)	$X_{6,6} \subset \mathbb{P}(2,2,2,3,3)$	$\mathbb{Z}_3^2 \times \mathbb{Z}_2$	$9A_1$

Table 4. Examples of singular K3 surfaces with the action of a finite abelian group

Recall that a Kummer K3 surface is the minimal resolution of a quotient of an abelian surface A by the multiplication by -1. In the next example, we show the existence of a Kummer K3 surface with the action of the group $H = \mathbb{Z}_4 \times \mathbb{Z}_2^3$.

Example 16. Let E_1 be an elliptic curve with a faithful action of $H' = \mathbb{Z}_2^3$, and E_2 be an elliptic curve with a faithful action of $H'' = \mathbb{Z}_2 \times \mathbb{Z}_4$ (so that the j-invariant of E_2 is equal to 1728). Note that H' acts on the set of 2-torsion points $\{a_1, a_2, a_3, a_4\}$ of E_1 transitively, while H'' acts on the set of 2-torsion points $\{b_1, b_2, b_3, b_4\}$ of E_2 with two orbits of cardinality 2, say, it interchanges b_1 with b_3 and b_2 with b_4. Consider an abelian surface $A = E_1 \times E_2$, which admits the action of $H' \times H'' = \mathbb{Z}_2^4 \times \mathbb{Z}_4$. It follows that $H' \times H''$ has 2 orbits of cardinality 8. Consider the quotient $S = A/\sigma$ where $\sigma = (\sigma_1, \sigma_2)$, and σ_i is the multiplication by -1 on E_i. Then S is a K3 surface with 16 singular points of type A_1. From the construction it follows that S admits a faithful action of $H = (H' \times H'')/(\mathbb{Z}_2) = \mathbb{Z}_2^3 \times \mathbb{Z}_4$. One checks that $\rho(S') = 20$ where S' is the minimal resolution of S.

Example 17. Similarly to Example 16, one can construct a Kummer K3 surface S with $16\, A_1$ singularities that admits a faithful action of $\mathbb{Z}_2^5$. Note that all the singular points of S lie in the same H-orbit in this case. As in the previous case, the Picard rank of the minimal resolution S' of S is equal to 20.

5.4 K3 surfaces: lattices

Let S be a smooth projective K3 surface. Consider the second cohomology group $\mathrm{H}^2(S,\mathbb{Z})$ as a lattice Λ_{K3} endowed with the cup product. It is well known that $\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ where U is the hyperbolic lattice, E_8 is the unique positive-definite, even, unimodular lattice of rank 8, and $E_8(-1)$ means that we multiply the Gram matrix of E_8 by (-1). Recall that on a K3 surface S we have $\mathrm{Pic}(S) = \mathrm{NS}(S)$, and

$$\mathrm{NS}(S) = \{x \in \mathrm{H}^2(S,\mathbb{Z}) \mid x \cdot \omega_S = 0\},$$

where ω_S is (the class of) a non-zero holomorphic 2-form on S. The transcendental lattice $T(S)$ is defined as follows:

$$T_S = \mathrm{NS}(S)^{\perp}.$$

It follows that the sublattice

$$\mathrm{NS}(S) \oplus T_S \subset \mathrm{H}^2(S,\mathbb{Z})$$

has finite index.

Example 18. For a Fermat quartic surface $S_4 \subset \mathbb{P}^3$, according to (Huybrechts, 2016, 3.2.6), one has

$$\mathrm{NS}(S) = E_8(-1)^{\oplus 2} \oplus U \oplus \mathbb{Z}(-8) \oplus \mathbb{Z}(-8), \qquad T_S = \mathbb{Z}(8) \oplus \mathbb{Z}(8).$$

In particular, one has $\rho(S_4) = 20$.

Lemma 20. Let S be a smooth K3 surface with a faithful action of finite group H. If the action of H on S is non-symplectic, then $\mathrm{NS}(S)^H = \mathrm{H}^2(S,\mathbb{Z})^H$.

Proof. Consider an element $x \in \mathrm{H}^2(S,\mathbb{Z})^H$. Consider its decomposition $x = x_{2,0} + x_{1,1} + x_{0,2}$ as an element of $\mathrm{H}^2(S,\mathbb{C})$ where $x_{2,0} \in \mathrm{H}^{2,0}(S)$, $x_{1,1} \in \mathrm{H}^{1,1}(S)$, and $x_{0,2} \in \mathrm{H}^{0,2}(S)$. Then for a non-symplectic element $h \in H$ we have

$$h^*(x) = h^*(x_{2,0}) + h^*(x_{1,1}) + h^*(x_{0,2}) = x_{2,0} + x_{1,1} + x_{0,2} = x.$$

Since the element h acts on the generator of $\mathrm{H}^{2,0}(S)$ (resp., of $\mathrm{H}^{2,0}(S)$) non-trivially, it follows that $x_{2,0} = 0$ (resp., $x_{0,2} = 0$). Hence $x \in \mathrm{H}^2(S,\mathbb{Z}) \cap \mathrm{H}^{1,1}(S) = \mathrm{NS}(S)$. Thus, $\mathrm{H}^2(S,\mathbb{Z})^H \subset \mathrm{NS}(S)^H$, and the result follows. □

Using Corollary 13, we obtain the following.

Corollary 19. For the groups (1)–(6) as in Theorem 19 we have $\mathrm{NS}(S)^H = \mathrm{H}^2(S,\mathbb{Z})^H$.

The following proposition is crucial to our work.

Proposition 20. Let H be one of the groups (1)–(6) in Theorem 19, and let S be a smooth K3 surface with a faithful action of H. Let M be the intersection matrix on $\mathrm{H}^2(S,\mathbb{Z})^H$. Put $r = \mathrm{rk}\,\mathrm{H}^2(S,\mathbb{Z})^H$. Then one of the following cases holds.

1. $H = \mathbb{Z}_6^2 \times \mathbb{Z}_2,\ r = 1,\ M = (2)$,
2. $H = \mathbb{Z}_4^3,\ r = 1,\ M = (4)$,
3. $H = \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2,\ r = 2,\ M = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$,
4. $H = \mathbb{Z}_6 \times \mathbb{Z}_3^2,\ r = 2,\ M = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$,
5. $H = \mathbb{Z}_2^5,\ 1 \le r \le 5$,
6. $H = \mathbb{Z}_4 \times \mathbb{Z}_2^3,\ 2 \le r \le 6$.

The possible intersection matrices in the cases (5) and (6) are presented in the Appendix.

Proof. Follows from the database provided by Brandhorst and Hofmann (2023). □

Lemma 21. Assume that a finite abelian group H faithfully acts on a K3 surface S with at worst du Val singularities. Let $f\colon \widetilde{S} \to S$ is the minimal resolution which is automatically H-equivariant. Put $r = \operatorname{rk} \mathrm{H}^2(S,\mathbb{Z})^H$. Then the following holds.

1. If $r = 1$, then S is smooth.
2. If $r = 2$, then the singular points of S form one H-orbit.

Corollary 21. Let H be a group isomorphic to $\mathbb{Z}_4^3$ or $\mathbb{Z}_6^2 \times \mathbb{Z}_2$. Then any K3 surface with a faithful action of H is smooth.

Proof. Follows from Lemma 21 and Proposition 20. □

Corollary 22. Let H be a group isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_6 \times \mathbb{Z}_3^2$ faithfully acting on a K3 surface S. Let $f\colon S' \to S$ be the minimal resolution. Then H acts on the set of f-exceptional (-2)-curves transitively. In particular, singularities on S form one H-orbit, and they can be only of type A_1 or A_2.

Remark 23. Let S be a K3 surface with du Val singularities endowed with an action of a finite abelian group H. Let $f\colon \widetilde{S} \to S$ be its minimal resolution. Note that f is automatically H-equivariant. Then

$$\mathrm{NS}(\widetilde{S})_{\mathbb{Q}} = f^*\mathrm{NS}(S)_{\mathbb{Q}} \oplus V_{\mathbb{Q}}$$

where $V = \langle E_i \rangle$ is a subgroup in $\mathrm{NS}(\widetilde{S})$ spanned by the f-exceptional (-2)-curves E_i, and $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$. However, it is not true that

$$\mathrm{NS}(\widetilde{S}) = f^*\mathrm{NS}(S) \oplus V.$$

Indeed, otherwise we would have

$$\mathrm{NS}(\widetilde{S})^H = f^*\mathrm{NS}(S)^H \oplus V^H,$$

which is not the case, as the following example shows.

Example 24. Consider a K3 surface S as in Example 15.(2) with the action of $H = \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$. Let $f\colon \widetilde{S} \to S$ be the minimal resolution. By Proposition 20 we have that $\Lambda = \mathrm{NS}(\widetilde{S})^H$ is the lattice with the Gram matrix

$$B_\Lambda = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Put

$$\Lambda' = f^*\mathrm{NS}(S)^H \oplus V^H$$

where $V = \langle E_i \rangle$ is a subgroup in $\mathrm{NS}(\widetilde{S})$ spanned by the f-exceptional (-2)-curves E_i. Then Λ' has the Gram matrix

$$B_{\Lambda'} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}.$$

This implies that $\rho^H(S) = 1$, and $\mathrm{NS}(S)^H = \mathbb{Z}[2A]$ where A is the restriction of $\mathscr{O}(1)$ in $\mathbb{P}(1,1,2,4)$ to S, so we have $A^2 = 1$. Then the lattice $2\Lambda = 2\mathrm{NS}(\widetilde{S})^H$, which has the Gram matrix

$$B_{2\Lambda} = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix},$$

is a sublattice of Λ'. We have the inclusions $2\Lambda \subset \Lambda' \subset \Lambda$.

Proposition 25. We have

$$\mu\mathrm{NS}(\widetilde{S}) \subset f^*\mathrm{NS}(S) \oplus V \subset \mathrm{NS}(\widetilde{S}).$$

where where $V = \langle E_i \rangle$ is a subgroup in $\mathrm{NS}(\widetilde{S})$ spanned by the f-exceptional (-2)-curves E_i, and μ is the index of $\mathrm{Cl}(S)$ in $\mathrm{NS}(S) = \mathrm{Pic}(S)$. Similarly, we have

$$\mu\mathrm{NS}(\widetilde{S})^H \subset f^*\mathrm{NS}(S)^H \oplus V^H \subset \mathrm{NS}(\widetilde{S})^H. \tag{22}$$

Consequently,

$$\det(\mathrm{NS}(\widetilde{S})^H) \quad | \quad \det(\mathrm{NS}(S)^H)\det V \quad | \quad \mu^{2\rho}\det(\mathrm{NS}(\widetilde{S})^H), \tag{23}$$

where ρ is the rank of $\mathrm{NS}(\widetilde{S})^H$.

Proof. Let $D \in \mathrm{NS}(\widetilde{S})^H$. Then $f^*f_*D = D + \sum a_iE_i$ where $E_i \in V$ and $a_i \in \mathbb{Z}[\frac{1}{\mu}]$. Hence, $\mu f_*D \in \mathrm{NS}(S)^H$, and μf^*f_*D belongs to $f^*\mathrm{NS}(S)^H$. Thus, $\mu(f^*f_*D - D) \in V$. This shows (22). Then (23) follows by taking the determinant of the lattices in (22). $\square$

5.5 Terminal singularities

In this section, we recall the classification of three-dimensional terminal singularities, cf. Mori (1985), Reid (1987). Let $x \in X$ be a germ of a three-dimensional terminal singularity. Then the singularity is isolated: $\mathrm{Sing}(X) = \{P\}$. The *index* of $x \in X$ is the minimal positive integer r such that rK_X is Cartier. If $r = 1$, then $x \in X$ is Gorenstein. In this case $P \in X$ is analytically isomorphic to a hypersurface singularity in $\mathbb{C}^4$ of multiplicity 2. Moreover, any Weil $\mathbb{Q}$-Cartier divisor D on $x \in X$ is Cartier. Also, in this case $x \in X$ is a compound du Val singularity, which means that its general hyperplane section H that contains P is a surface with a du Val singularity at x. We say that $x \in X$ has type cA, cD, or cE if $x \in H$ is a du Val singularity of type A, D, or E, respectively. If $r > 1$, then there is a cyclic étale outside P covering

$$\pi\colon \widetilde{x} \in \widetilde{X} \to X \ni x$$

of degree r such that $\widetilde{x} \in \widetilde{X}$ is a Gorenstein terminal singularity (or a smooth point). The map π is called the *index-one cover* of $x \in X$, and it is defined canonically.

Theorem 24 (Mori (1985); Reid (1987)). Let $x \in X$ be a three-dimensional terminal singularity of index $r > 1$. Then $x \in X$ is analytically isomorphic to the quotient $\{\varphi = 0\}/(\mathbb{Z}_r)$ of a hypersurface $\mathbb{C}^4$ defined by the equation

$$\varphi(x_1, \ldots, x_4) = 0,$$

where the group $\mathbb{Z}_r$ acts on $\mathbb{C}^4$ such that the coordinates x_i and the equation φ are semi-invariant. Moreover, up to an analytic $\mathbb{Z}_r$-equivariant coordinate change, the hypersurface $\varphi = 0$ and the $\mathbb{Z}_r$-action are described by one of the rows in the following table (where $\mathfrak{m}$ denotes the maximal ideal of $x \in X$):

Type	**Index**	**Equation** $\varphi(x_1,x_2,x_3,x_4)$	**Weights**
cA/r	$r \geq 1$	$x_1x_2 + \psi(x_3^r, x_4)$	$(1,-1,a,0;0)$, $(r,a) = 1$
$cAx/2$	$r = 2$	$x_1^2 + x_2^2 + \psi(x_3,x_4)$, $\psi \in \mathfrak{m}^4$	$(0,1,1,1;0)$
$cD/2$	$r = 2$	$x_4^2 + \psi(x_1,x_2,x_3)$, $\psi \in \mathfrak{m}^3$ with $x_1x_2x_3$ or $x_2^2x_3 \in \psi$	$(1,1,0,1;0)$
$cE/2$	$r = 2$	$x_4^2 + x_1^3 + \psi(x_2,x_3)x_1 + \theta(x_2,x_3)$, $\theta \notin \mathfrak{m}^5$	$(0,1,1,1;0)$
$cD/3$	$r = 3$	$x_4^2 + \psi(x_1,x_2,x_3)$ with $\psi_3 = x_1^3 + x_2^3 + x_3^3$ or $x_1^3 + x_2x_3^2$ or $x_1^3 + x_2^3$	$(1,2,2,0;0)$
$cAx/4$	$r = 4$	$x_1^2 + x_2^2 + \psi(x_3^2, x_4)$, $\psi \in \mathfrak{m}^3$	$(1,3,1,2;2)$

Table 5. Terminal singularities

5.5.1 Baskets of singularities

For a three-dimensonal terminal singularity $x \in X$ there is a deformation to $k \geq 1$ terminal cyclic quotient singularities $x_1, \dots, x_k$. The number $k = \mathrm{aw}(x \in X)$ is called the *axial weight* of $x \in X$. We may assume that the singularities x_i have type $\frac{1}{r_i}(1, -1, b_i)$ where $0 < b_i \leq r_i/2$. The collection $\{x_1, \dots, x_k\}$ is known as the *basket of singularities* of $x \in X$ and it can be written as

$$B(x \in X) = \left\{ n_i \times \frac{1}{r_i}(1, -1, b_i) \right\}.$$

By the basket of singularities of X, denoted by $B(X)$, we mean the union of all baskets of $x \in X$ for all non-Gorenstein singular points $x \in X$.

If $x \in X$ is a non-Gorenstein singularity of index r, then in its basket of cyclic quotient singularities all the points in the basket have index r, except in the case when $x \in X$ is of type $cAx/4$, in which case one of the points in the basket has index 4, and all the other points in the basket have index 2. Moreover, if $x \in X$ is not a quotient singularity itself, then in the basket of $x \in X$ there are at least two points.

5.5.2 Orbifold Riemann-Roch

By (Reid, 1987, 10.2), for a terminal threefold X and a Weil $\mathbb{Q}$-Cartier divisor D on it we have the following version of the Riemann-Roch formula:

$$\chi(\mathscr{O}_X(D)) = \chi(\mathscr{O}_X) + \frac{1}{12}D(D - K_X)(2D - K_X) + \frac{1}{12}D \cdot c_2(X) + \sum_Q c_Q(D) \qquad (25)$$

where for any cyclic quotient singularity Q we have

$$c_Q(D) = -i\frac{r^2 - 1}{12r} + \sum_{j=1}^{i-1} \frac{\overline{bj}(r - \overline{bj})}{2r}. \qquad (26)$$

Here r is the index of Q, the divisor D has type $i\frac{1}{r}(a, -a, 1)$ at Q, b satisfies $ab = 1 \bmod r$, and $\bar{\ }$ denotes the residue modulo r. For non-cyclic non-Gorenstein singularities, their contribution to the right-hand side of 25 is computed in terms of their basket of cyclic points. Moreover, by (Reid, 1987, 10.3) (see (Y. G. Prokhorov, 2021, Theorem 12.1.3) in the case of $G\mathbb{Q}$-Fano threefolds) one has

$$(-K_X) \cdot c_2(X) + \sum_Q (r - 1/r) = 24 \qquad (27)$$

Since $(-K_X) \cdot c_2(X) > 0$, we see that the number of non-Gorenstein singular points is at most 15.

5.5.3 Geometry of the flag $x \in S \subset X$

We start with the following lemma which is well-known to experts.

Lemma 28. Assume that X is a terminal threefold. Let $S \in |-K_X|$ be an anti-canonical element, and let x be a point which is singular on X. Then x is a singular point on S.

Proof. We may assume that $x \in S \subset X$ is a germ. The claim is clear if x is a Gorenstein point. Assume that x is non-Gorenstein of index $r > 1$. Consider the index one cover $\pi\colon \widetilde{X} \to X$ which is étale of degree r outside x. Assume that S is smooth at x. Put $\pi^{-1}(S) = \widetilde{S}$. Then $\pi|_{\widetilde{S}\setminus\widetilde{x}}\colon \widetilde{S} \setminus \{\widetilde{x}\} \to S \setminus \{x\}$ is étale outside P as well. Since by assumption S is smooth at P, it follows that $S \setminus \{x\}$ is simply-connected. Hence the cover $\pi|_{\widetilde{S}\setminus\{\widetilde{x}\}}\colon \widetilde{S} \setminus \{\widetilde{x}\} \to S \setminus \{x\}$ splits, so that $\widetilde{S} = \sum \widetilde{S}_i$, and $\widetilde{S}_i \cap \widetilde{S}_j = \{\widetilde{x}\}$ for $i \neq j$. However, since rS is Cartier, it follows that $r\widetilde{S}$ is Cartier as well, and hence Cohen-Macaulay (here we use the fact that three-dimensional terminal singularities are Cohen-Macaulay). Thus, $\widetilde{S} \setminus \{\widetilde{x}\}$ should be connected, which is not the case. This leads to a contradiction, which shows that S is singular at x. The result follows. □

Remark 26. Let $x \in S$ be a germ of a du Val singularity. Let $\pi_1^{ab}(S \setminus \{x\})$ be the abelianization of the local fundamental group. Then there are the following possibilities for $\pi_1^{ab}(S \setminus \{x\})$ according to the type of singularity:

1. $\mathbb{Z}_{n+1}$ for type A_n,
2. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for type D_n for even $n \geq 4$,
3. $\mathbb{Z}_2$ for type D_n for odd $n \geq 5$,
4. $\mathbb{Z}_3$ for type E_6,
5. $\mathbb{Z}_2$ for type E_7,
6. 0 for type E_8.

Assume there exists a finite morphism $\widetilde{x} \in \widetilde{S} \to S \ni x$ such that the induced map $\pi|_{\widetilde{S}\setminus\{\widetilde{x}\}}\colon \widetilde{S} \setminus \{\widetilde{x}\} \to S \setminus \{x\}$ is a non-split cyclic étale covering of degree k. Then $\pi_{\widetilde{S}\setminus\{\widetilde{x}\}}$ corresponds to a subgroup H of $\pi_1^{\mathrm{ab}}(S \setminus \{x\})$ such that $\pi_1^{\mathrm{ab}}(S \setminus \{x\})/H = \mathbb{Z}_k$. In particular, if $x \in S \subset X$ where $x \in X$ is a non-Gorenstein threefold terminal point of index r and $S \in |-K_X|$ has du Val singularities, then $\pi|_{\widetilde{S}\setminus\{\widetilde{x}\}}$ is non-split (see the proof of Lemma 28) cyclic étale covering of degree r, hence r divides $|\pi_1^{ab}(S \setminus \{x\})|$.

Let $x \in X$ be the germ of a terminal singularity, and let $S \in |-K_X|$ be an anti-canonical element. Assume that the pair (X, S) is plt. Let $\pi\colon \widetilde{x} \in \widetilde{X} \to X \ni x$ be the index 1 cover. Note that π induces a cover $\pi|_{\widetilde{S}}\colon \widetilde{x} \in \widetilde{S} \to S \ni x$. There exists the following diagram:

$$
\begin{array}{ccc}
\widetilde{x} \in \widetilde{X} & \xrightarrow{\ \pi\ } & X \ni x \\
\uparrow & & \uparrow \\
\widetilde{x} \in \widetilde{S} & \xrightarrow{\ \pi|_{\widetilde{S}}\ } & S \ni x
\end{array}
\tag{29}
$$

We consider two cases: when $x \in X$ is a cyclic quotient singularity, so $(\widetilde{x} \in \widetilde{X}) \simeq (0 \in \mathbb{C}^3)$, and when $x \in X$ is not a cyclic quotient singularity, so $(\widetilde{x} \in \widetilde{X}) \subset (0 \in \mathbb{C}^4)$. Note that in the latter case $\widetilde{S}$ is a Cartier divisor on $\widetilde{X} \subset \mathbb{C}^4$, hence $\widetilde{S}$ is singular at $\widetilde{x}$.

Proposition 27. Assume that $x \in X$ is a cyclic quotient singularity of index $r \geq 1$. If $\widetilde{x} \in \widetilde{S}$ is smooth then $x \in S$ has type A_{r-1}.

Proof. By assumption, $\widetilde{x} \in \widetilde{S}$ is smooth, so $(\widetilde{x} \in \widetilde{S}) \simeq (0 \in \mathbb{C}^2)$. Hence the non-split degree r cyclic covering $\pi|_{\widetilde{S}\setminus\{\widetilde{x}\}} \colon \widetilde{S} \setminus \{\widetilde{x}\} \to S \setminus \{x\}$ is a universal covering. It follows that $\pi_1(S \setminus \{x\})$ is a group of order r. Since $\pi|_{\widetilde{X}\setminus\{\widetilde{x}\}}$ is a cyclic covering, the group of deck transformations of $\pi|_{\widetilde{X}\setminus\{\widetilde{x}\}}$ is cyclic of order r. Thus, the group of deck transformations of $\pi|_{\widetilde{S}\setminus\{\widetilde{x}\}}$ contains an element of order r. Since $|\pi_1(S \setminus \{x\})| = r$, it follows that $\pi_1(S \setminus \{x\}) = \mathbb{Z}_r$. This implies that $x \in S$ is a singular point of type A_{r-1}. □

5.5.4 Group action on a terminal singularity

Let X be a $G\mathbb{Q}$-Fano threefold where G is a finite abelian group. Let $x \in X$ be a germ of a terminal singularity. Assume there exists a G-invariant element $S \in |-K_X|$ which is a K3 surface with du Val singularities. Then there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_m \longrightarrow G \longrightarrow H \longrightarrow 0 \tag{30}$$

where H faithfully acts on S, and $\mathbb{Z}_m$ faithfully acts in the normal bundle to S in X for some $m \geq 1$. Let G_x and H_x be the stabilizers of x in G and H, respectively. Thus, we obtain the exact sequence

$$0 \longrightarrow \mathbb{Z}_m \longrightarrow G_x \longrightarrow H_x \longrightarrow 0 \tag{31}$$

Theorem 32 ((Loginov, 2024, Theorem 7.3)). Let $x \in X$ be a germ of a threefold terminal singularity and let $G_x \subset \mathrm{Aut}(x \in X)$ be a finite abelian subgroup. Then either $\mathfrak{r}(G_x) \leq 3$, or

$$G_x = \mathbb{Z}_2^2 \times \mathbb{Z}_2 n \times \mathbb{Z}_2 m$$

for $n, m \geq 1$. Moreover, in the latter case $x \in X$ is a Gorenstein singularity of type cA. In particular, in both cases G is of product type.

The diagram (29) induces the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_r & \longrightarrow & \widetilde{G}_x & \longrightarrow & G_x & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_r & \longrightarrow & \widetilde{H}_x & \longrightarrow & H_x & \longrightarrow & 0 \end{array} \tag{33}$$

where $\widetilde{G}_x$ is the lifting of G_x, and $\widetilde{H}_x$ is the lifting of H_x. This means that $\widetilde{G}_x$ faithfully acts on $\widetilde{x} \in \widetilde{X}$, and $\widetilde{H_x}$ faithfully acts on $\widetilde{x} \in \widetilde{S}$.

Proposition 28. Assume that

- either $r > 2$,
- or G_x is a 2-group.

Then the lifting $\widetilde{G}_x$ is abelian.

Proof. The first claim is (Loginov, 2024, Proposition 7.13). The second claim follows from the proof of (Loginov, 2024, Proposition 7.18). □

Proposition 29. Assume that $x \in X$ is a cyclic quotient singularity of index $r \geq 1$. If $\widetilde{x} \in \widetilde{S}$ is singular then G is of product type.

Proof. Assume that $\widetilde{x} \in \widetilde{S}$ is singular. Since $T_{\widetilde{x}}\widetilde{S} = T_{\widetilde{x}}\widetilde{X} = \mathbb{C}^3$, by Lemma 5 we know that $\widetilde{G}_x = \widetilde{H}_x$, and hence $G_x = H_x$. It follows that in the exact sequences (**??**) and (30) we have $m = 1$. Thus, we have $G = H$. Therefore G is of product type by Remark 12. □

Corollary 30. Assume that $x \in X$ is a cyclic quotient singularity of index $r \geq 1$. Then either G is of product type, or $\widetilde{x} \in \widetilde{S}$ is smooth and $x \in S$ is a singularity of type A_{r-1}.

Proof. Follows from Proposition 27 and Proposition 29. □

Remark 31*.* Assume that $x \in X$ is a cyclic quotient singularity of index $r \geq 1$. Assume that $(\widetilde{x} \in \widetilde{S})$ is smooth, that is $(\widetilde{x} \in \widetilde{S}) \simeq (0 \in \mathbb{C}^2)$. Then we have $\widetilde{G}_x = \widetilde{H}_x \times \mathbb{Z}_k$ for some $k \geq 1$. In particular, there exists a homomorphism $\widetilde{H}_x \to \widetilde{G}_x$ which is inverse to the map $\widetilde{G}_x \to \widetilde{H}_x$ as in (33).

Proposition 32. Assume that $x \in X$ is a non-Gorenstein singularity which is not a cyclic quotient singularity. Then the singular point $x \in S$ cannot be of type A_1 or A_2. In particular, the action of H on the set of (-2)-curves on the minimal resolution S' of S cannot be transitive.

Proof. By Lemma 28, the point $x \in S$ is singular. Since $x \in X$ is not a cyclic quotient singularity, we have that $\widetilde{x} \in \widetilde{X}$ is singular. Since $\widetilde{S}$ is Cartier at the point $\widetilde{x}$ in $\widetilde{X}$, we see that the point $\widetilde{x} \in \widetilde{S}$ is singular as well. Hence the map $\pi|_{\widetilde{S}\setminus\{\widetilde{x}\}} \colon \widetilde{S} \setminus \{\widetilde{x}\} \to S \setminus \{x\}$ is a non-trivial covering which is not universal. Thus, $\widetilde{x} \in \widetilde{S}$ cannot be of type A_1 or A_2, because in these cases $S \setminus \{x\}$ does not admit such a covering, cf. Remark 26. The last claim follows from looking at the diagrams of exceptional curves on the minimal resolution S' of S. □

5.6 Case $h^0(-K_X) \geq 2$

Let X be a $G\mathbb{Q}$-Fano threefold where G is a finite abelian group. Assume that $h^0(X, \mathscr{O}(-K_X)) \geq 2$, and that all the G-invariant elements in $|-K_X|$ are K3 surfaces with at worst canonical singularities. Let S_1 and S_2 be two such surfaces. For $i = 1, 2$ consider an exact sequence

$$0 \longrightarrow C_i \longrightarrow G \longrightarrow G_i \longrightarrow 0 \qquad (34)$$

where G_1 faithfully acts on S_i, and C_i fixes S_i pointwise. Moreover, C_i is a cyclic group that faithfully acts in the normal bundle to S_i in X.

Lemma 35. We have that C_1 (resp. C_2) faithfully acts on S_2 (resp. S_1), and this action is purely non-symplectic. In particular, we have $C_1 \cap C_2 = \{\mathrm{id}\}$, and the maps $C_1 \hookrightarrow G_2$ and $C_2 \hookrightarrow G_1$ induced by the exact sequences (34) are injective.

Proof. Assume that C_1 does not act faithfully on S_2. Then there is a non-trivial element $g \in C_1$ acting trivially on S_2. The fixed locus of g contains $S_1 \cup S_2$, hence is singular along a curve $S_1 \cap S_2$. This is impossible by Lemma 6. Finally, expressing the volume form on S_2 in local coordinates at the general point $x \in S_1 \cap S_2$, we see that the action of C_1 on S_2 is purely non-symplectic. The claim for the action of C_2 on S_1 follows by symmetry. □

From Lemma 35 and Corollary 13 we immediately obtain

Corollary 33. Fix $i \in \{1, 2\}$. Put $C_i = \mathbb{Z}_{n_i}$. Then $n_i \in \{1, 2, 3, 4, 6, 8\}$.

Lemma 36. Fix $i \in \{1, 2\}$. Consider an exact sequence

$$0 \longrightarrow C_i \longrightarrow G \longrightarrow G_i \longrightarrow 0$$

as in (34) where $C_i = \mathbb{Z}_{n_i}$, and $n_i \in \{1, 2, 3, 4, 6, 8\}$. If this exact sequence does not split, then G is of product type.

Proof. It is enough to prove the result for $i = 1$. The group G is isomorphic to the direct product of $G_{p_1} \times \cdots \times G_{p_k}$, where G_{p_i} is an abelian group which fits in a short exact sequence of the form

$$0 \longrightarrow C_{1,p_i} \longrightarrow G_{p_i} \longrightarrow G_{1,p_i} \longrightarrow 0$$

where C_{1,p_i} and G_{1,p_i} are p_i-Sylow subgroups of C_1 and G_1, respectively. Conversely, any direct product $G_{p_1} \times \cdots \times G_{p_k}$ is a possible isomorphism class for G. We will proceed as follows.

1. We fix an isomorphism class for G_1, among those in Theorem 19.
2. For each $n \in \{1, 2, 3, 4, 6, 8\}$, we use Theorem 9 to compute all possible classes of p-Sylow subgroups of G.
3. We deduce the possible isomorphism classes for G.

Applying systematically this method, we obtain the following. We have $C_1 = \mathbb{Z}_{n_1}$. Put $n = n_1$. Assume that G is not isomorphic to $\mathbb{Z}_n \times G_1$ and that $\mathfrak{r}(G) > 3$. Then for $n \in \{1,2,3,4,6,8\}$ we have the following possibilities.

- $G_1 = \mathbb{Z}_4^3$. Then we have the following possibilities
  1. $n = 4$, $G = \mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$,
  2. $n = 8$, $G = \mathbb{Z}_16 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$.
- $G_1 = \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$. Then we have the following
  1. $n = 4$, $G = \mathbb{Z}_16 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$, or $G = \mathbb{Z}_8^2 \times \mathbb{Z}_2^2$,
  2. $n = 8$, $G = \mathbb{Z}_32 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$, or $G = \mathbb{Z}_16 \times \mathbb{Z}_8 \times \mathbb{Z}_2^2$, or $G = \mathbb{Z}_16 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$.
- $G_1 = \mathbb{Z}_4 \times \mathbb{Z}_2^3$. Then
  1. $n = 2$, $G = \mathbb{Z}_8 \times \mathbb{Z}_2^3$, or $G = \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$,
  2. $n = 4$, $G = \mathbb{Z}_16 \times \mathbb{Z}_2^3$, or $G = \mathbb{Z}_8 \times \mathbb{Z}_2^4$, or $G = \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$,
  3. $n = 6$, $G = \mathbb{Z}_24 \times \mathbb{Z}_2^3$,
  4. $n = 8$, $G = \mathbb{Z}_32 \times \mathbb{Z}_2^3$, or $G = \mathbb{Z}_16 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$, or $G = \mathbb{Z}_16 \times \mathbb{Z}_2^4$.
- $G_1 = \mathbb{Z}_2 \times \mathbb{Z}_3^3$. In this case we always have $\mathfrak{r}(G) \leq 3$.
- $G_1 = \mathbb{Z}_3^2 \times \mathbb{Z}_2^3$. We obtain that $\mathfrak{r}(G) \leq 3$, or that G is isomorphic to $\mathbb{Z}_n \times G_1$.
- $G_1 = \mathbb{Z}_2^5$. $G = \mathbb{Z}_2^i \times \mathbb{Z}_2^4$ with $2 \leq i \leq 4$, or $G = \mathbb{Z}_12 \times \mathbb{Z}_2^4$.

Among all these possibilities, we see that if G is not isomorphic to $\mathbb{Z}_n \times G_1$, then it is of product type. □

Theorem 37. Assume that X is a $G\mathbb{Q}$-Fano threefold with $h^0(-K_X) \geq 2$ where G is a finite abelian group. If G is of K3 type and not of product type then G is isomorphic to one of the following groups:

1. $\mathbb{Z}_4^4$,
2. $\mathbb{Z}_8^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$,
3. $\mathbb{Z}_6^2 \times \mathbb{Z}_3^2$,
4. $\mathbb{Z}_6^3 \times \mathbb{Z}_2$.

Proof. By Lemma 36, we may assume that $G = \mathbb{Z}_n \times G_1$, where G_1 is one of the groups (1)–(6) as in Theorem 19. The group G also fits into the exact sequence

$$0 \longrightarrow C_2 \longrightarrow G \longrightarrow G_2 \longrightarrow 0$$

Moreover, we have that $\mathbb{Z}_n \subset G_2$, and $C_2 \subset G_1$. We will proceed in the following way.

1. For a given group G_1 from Theorem 19, we deduce the possibilities for C_2.
2. For a given pair (G_1, C_2), we find all the possibilities for n.

We obtain the following possible configurations.

- $G_1 = \mathbb{Z}_4^3$.

C_2	$\{\mathrm{Id}\}$	$\mathbb{Z}_2$	$\mathbb{Z}_4$
n	1	1,2	1,2,4

The group G is isomorphic to a subgroup of $\mathbb{Z}_4^4$.

- $G_1 = \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$.

C_2	$\{\mathrm{Id}\}$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_8$
n	1	1,2	1,2,4	1,2,4,8

The group G is isomorphic to a subgroup of $\mathbb{Z}_8^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$.

- $G_1 = \mathbb{Z}_4 \times \mathbb{Z}_2^3$.

C_2	$\{\mathrm{Id}\}$	$\mathbb{Z}_2$	$\mathbb{Z}_4$
n	1	1,2	1,2,4

The group G is isomorphic to a subgroup of $\mathbb{Z}_4^2 \times \mathbb{Z}_2^3$, which is of product type.

- $G_1 = \mathbb{Z}_6 \times \mathbb{Z}_3^2$.

C_2	$\{\mathrm{Id}\}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_6$
n	1	1,2	1,3	1,2,3,6

The group G is isomorphic to a subgroup of $\mathbb{Z}_6^2 \times \mathbb{Z}_3^3$.

- $G_1 = \mathbb{Z}_6^2 \times \mathbb{Z}_2$.

C_2	$\{\mathrm{Id}\}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_6$
n	1	1,2	1,3	1,2,3,6

The group G is isomorphic to a subgroup of $\mathbb{Z}_6^3 \times \mathbb{Z}_2$.

- $G_1 = \mathbb{Z}_2^5$.

C_2	$\{\mathrm{Id}\}$	$\mathbb{Z}_2$
n	1	1,2

The group G is isomorphic to a subgroup of $\mathbb{Z}_2^6$, which is of product type.

$\square$

5.7 Orbits of non-Gorenstein points

Let G be a finite abelian group, and ket X be a 3-dimensional $G\mathbb{Q}$-Fano variety. Assume that the set of non-Gorenstein singularities of X is non-empty. Let us denote the set of all non-Gorenstein points of X as follows:

$$k_1 \times P_1, \quad k_2 \times P_2, \quad \ldots, \quad k_l \times P_l, \qquad k_i \geq 1 \tag{38}$$

for $l \geq 1$ where each $P_i \in X$ is a germ of a terminal non-Gorenstein singularity of index $r_i > 1$, and $k_i \times P_i$ means that we have exactly k_i singular points of X locally analytically.isomorphic to $P_i \in X$. In particular, $P_i \in X$ and $P_j \in X$ are not locally analytically isomorphic for $i \neq j$. Hence, each set $\{k_i \times P_i\}$ for $1 \leq i \leq l$ splits into G-orbits.

Assume that the pair (X,S) is plt where $S \in |-K_X|$ is G-invariant element. Then S is a K3 surface with at worst canonical singularities. We have the following exact sequence:

$$0 \longrightarrow \mathbb{Z}_m \longrightarrow G \longrightarrow H \longrightarrow 0 \tag{39}$$

where H faithfully acts on S, and $m \geq 1$. By Lemma 28, we see that S has at least $k_1 + \ldots + k_l$ du Val singularities that correspond to non-Gorenstein singularities on X. We denote by $f\colon \widetilde{S} \to S$ the H-equivariant minimal resolution of S, so that $\widetilde{S}$ is a smooth K3 surface with a faithful action of H. We assume that H is one of the groups (1)–(6) as in Proposition 20. We examine them case by case.

Lemma 40. The group H in (39) cannot be isomorphic either to $\mathbb{Z}_6^2 \times \mathbb{Z}_2$ or to $\mathbb{Z}_4^3$.

Proof. By Corollary 21, in this case S is a smooth K3 surface. However, this contradicts to the assumption $l \geq 1$ and Lemma 28. □

Lemma 41. If $H = \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ and G is not of product type, then $l = 1$ and k_1 is even. Morever, the action of H (as well as G) on the set $\{k_i \times P_1\}$ is transitive. In particular, k_1 divides 64.

Proof. By Corollary 22, there exists at most one H-orbit of singular points of S. By Lemma 28 it follows that there exists at most one G-orbit of non-Gorenstein singular points on X. Thus, we have $l = 1$, and the action of G on the set $\{k_i \times P_1\}$ is transitive. If k_1 is odd, then G has a fixed point, so by Theorem 32 we have that G is of product type. The last claim follows from the orbit-stabilizer theorem. □

Lemma 42. If $H = \mathbb{Z}_6 \times \mathbb{Z}_3^2$ and G is not of product type, then $l = 1$, and k_1 is divisible by 3. Moreover, the action of H (as well as G) on the set $\{k_i \times P_1\}$ is transitive. In particular, k_1 divides 54.

Proof. As in the proof of Lemma 41, by Lemma 28 and Corollary 22 we have $l = 1$, and the action of G on the set $\{k_i \times P_1\}$ is transitive. If k_1 is not divisible by 3, then G_3 has a fixed point. We deduce from Theorem 32 that G_3 is of rank 3. For $p \neq 3$, since $\mathfrak{r}(H_p) \leq 1$, we have $\mathfrak{r}(G_p) \leq 2$ for $p \neq 3$. We deduce that $\mathfrak{r}(G) \leq 3$. Thus G is of product type. The last claim follows from the orbit-stabilizer theorem. □

Lemma 43. If $H = \mathbb{Z}_2^5$ and G is not of product type, then k_i is divisible by 8 for all $i \in \{1, \ldots, l\}$.

Proof. If there exists $i \in \{1, \ldots, l\}$ such that k_i is not divisible by 8, then G_2 has an orbit of length $1, 2$, or 4. But since $H = \mathbb{Z}_2^5$, the group G_2 is of the form $\mathbb{Z}_2^k \times \mathbb{Z}_2^5$, for some $k \geq 1$, or $\mathbb{Z}_2^k \times \mathbb{Z}_2^4$, for some $k \geq 1$. Hence, all subgroups of G_2 of index 1 or 2 have rank at least 4, and Theorem 32 implies that G_2 cannot have an orbit of length 1 or 2. Assume there exists an orbit of length 4. Then G_2 has a subgroup of rank 3 or 4 fixing a point, and the latter is excluded by the same result. So G_2 has a subgroup of index 4 and rank 3, which implies that G_2 is of the form $\mathbb{Z}_2^k \times \mathbb{Z}_2^4$, for some $k \geq 1$. But then G is of product type. □

Lemma 44. If $H = \mathbb{Z}_4 \times \mathbb{Z}_2^3$ and G is not of product type, then k_i is divisible by 4 for all $i \in \{1, \ldots, l\}$.

Proof. If there exists $i \in \{1, \ldots, l\}$ such that k_i is not divisible by 4, then G_2 has an orbit of length 1 or 2. The first case is excluded by Theorem 32, since $r(G_2) \geq r(H) = 4$. If G_2 has a subgroup G_2' of index 2 fixing a point, then G_2' is of rank 3, and we deduce that either $r(G_2) \leq 3$, and hence G is of product type, or $G_2 = \mathbb{Z}_2^k \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$ for $k \geq 1$, or $G_2 = \mathbb{Z}_2^k \times \mathbb{Z}_2^3$, with $k \geq 2$. In all cases, the group G is of product type. □

We obtain the following.

Corollary 34. Either G is of product type, or $\gcd(k_1, \ldots, k_l)$ is divisible either by 2 or by 3, where k_i are as in (38). In particular, if $\gcd(k_1, \ldots, k_l) = 1$ then G is of product type.

After applying Corollary 34 as well as Lemmas 40–44 to all the Fano threefolds of Fano index 1 and Fano genus -1, which we went through using the Graded Ring Database, and taking off the baskets consisting only of singularities of type $1/2(1,1,1)$ which will be treated in Section 5.8, we end up with the following result.

Proposition 35. Assume that $h^0(-K_X) = 1$. If G is not of product type, then its basket is among the following possibilities.

Basket of singularities of X	**Possibilities for H**
$9 \times 1/2(1,1,1)$	$\mathbb{Z}_6 \times \mathbb{Z}_3^2$
$12 \times 1/2(1,1,1)$	$\mathbb{Z}_4 \times \mathbb{Z}_2^3$
$2 \times 1/10(3,7,1)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$2 \times 1/11(4,7,1)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$

Basket of singularities of X	**Possibilities for H**
$6\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_6\times\mathbb{Z}_3^2$
$2\times 1/9(2,7,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$6\times 1/2(1,1,1), 2\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$4\times 1/2(1,1,1), 4\times 1/3(1,2,1)$	$\mathbb{Z}_4\times\mathbb{Z}_2^3$
$4\times 1/5(2,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2^3$
$4\times 1/2(1,1,1), 4\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2^3$
$2\times 1/11(3,8,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$8\times 1/3(1,2,1)$	$\mathbb{Z}_2^5$, $\mathbb{Z}_4\times\mathbb{Z}_2^3$, $\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$3\times 1/7(2,5,1)$	$\mathbb{Z}_6\times\mathbb{Z}_3^2$
$6\times 1/2(1,1,1), 3\times 1/4(1,3,1)$	$\mathbb{Z}_6\times\mathbb{Z}_3^2$
$2\times 1/11(2,9,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$8\times 1/2(1,1,1), 2\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$10\times 1/2(1,1,1), 2\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$8\times 1/2(1,1,1), 4\times 1/3(1,2,1)$	$\mathbb{Z}_4\times\mathbb{Z}_2^3$

Table 6. Possible baskets of singularities on X and corresponding groups H

5.8 Case $h^0(-K_X)=1$ with half-points

To illustrate our approach, we treat the case when the non-Gorenstein points of X have type $\frac{1}{2}(1,1,1)$. The main goal of this section is to prove the following

Theorem 45. Let X be a $G\mathbb{Q}$-Fano threefold where G is a finite abelian group. Assume that $h^0(-K_X)=1$. Also, assume that the non-Gorenstein locus of X consists only of points of type $\frac{1}{2}(1,1,1)$. Then G is of product type.

Proposition 36. Assume that all the terminal non-Gorenstein points of X have type $1/2(1,1,1)$. Then $9\leq N\leq 15$.

Proof. Since $h^0(X,-K_X)=1$, we see that X is non-Gorenstein. For $D=-K_X$ and singular points of type $\frac{1}{2}(1,1,1)$ we have $c_Q=-1/8$, so if there are N such points, in the formula (25) they give a contribution $-N/8$. From (27) it follows that

$$(-K_X)\cdot c_2(X)=24-\frac{3N}{2}. \qquad (46)$$

We have

$$1 = h^0(\mathscr{O}_X(-K_X)) = 3 + \frac{1}{2}(-K_X)^3 - \frac{N}{4} \tag{47}$$

Using $(-K_X)^3 \geq 1/2$ we obtain $N \geq 9$. Since by the above we have $N \leq 15$, the result follows. □

We work in the setting of Section 5.7. In particular, we assume that H is one of the groups (1)–(6) as in Proposition 20. By Corollary 34, we see that in the cases $N = 11, 13$ the group G is of product type. In the cases $N = 10, 14, 15$ by Lemmas 40–44 we see that the group G is of product type as well. It remains to deal with the cases $N = 9, 12$.

Proposition 37. If $N = 9$ then G is of product type.

Proof. By Lemmas 40–44 we see that $H = \mathbb{Z}_6 \times \mathbb{Z}_3^2$. Moreover, the 9 singular points of S that correspond to the 9 singular points on X of type $1/2(1,1,1)$ form one G-orbit of length 9. Let $f\colon S' \to S$ be a H-equivariant minimal resolution of S. By Corollary 22, the surface S has singularities of type A_1 or A_2. In both cases there is an f-exceptional G-invariant curve of self-intersection -18 on S'. The index μ of $\mathrm{Cl}(S)$ in $\mathrm{NS}(S)$ is equals either 2 or 3. By Proposition 25, we have

$$\mu \mathrm{NS}(S')^H \subset f^*(\mathrm{NS}(S)^H) \oplus V \subset \mathrm{NS}(S')^H. \tag{48}$$

Assume that $\mu = 2$. By Proposition 20 we have

$$\left(2\mathrm{NS}(S')^H, \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix}\right) \subset \left(f^*(\mathrm{NS}(S)^H) \oplus V, \begin{pmatrix} a & 0 \\ 0 & -18 \end{pmatrix}\right) \subset \left(\mathrm{NS}(S')^H, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}\right).$$

Note that $a = 6a'$, since the values of the intersection form on $\mathrm{NS}(S')^H$ is divisible by 6. By (23) we have that $-108a'$ divides -144 for $a' \geq 1$ which is a contradiction.

Assume that $\mu = 3$. Analogously to the previous case, we have

$$\left(2\mathrm{NS}(S')^H, \begin{pmatrix} 0 & 27 \\ 27 & 0 \end{pmatrix}\right) \subset \left(f^*(\mathrm{NS}(S)^H) \oplus V, \begin{pmatrix} a & 0 \\ 0 & -18 \end{pmatrix}\right) \subset \left(\mathrm{NS}(S')^H, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}\right).$$

Then $-18a$ divides -27^2, which is a contradiction. This shows that G is of product type. □

The fact that the case $N = 12$ cannot realize follows from the following.

Proposition 38. Let X be a $G\mathbb{Q}$-Fano threefold such that all non-Gorenstein singular points of X are points of type $1/2(1,1,1)$. Assume that $S \in |-K_X|$ is a G-invariant K3 surface with at worst du Val singularities. Assume that

1. $H = \mathbb{Z}_4 \times \mathbb{Z}_2^3$ where H is as in (34),
2. x belongs to a G-orbit of length 4.

Then G is of product type.

Proof. By Corollary 30 and using its notation, we may assume that $\widetilde{x} \in \widetilde{S}$ is smooth. Hence, in the notation of diagram (33), we see that $\mathfrak{r}(\widetilde{H}_x) \leq 2$ and so $\mathfrak{r}(H_x) \leq 2$. Let $\Sigma = \{x = x_1, x_2, x_3, x_4\}$ be the G-orbit of x (or, equivalently, its H-orbit). Consider the following exact sequence

$$0 \longrightarrow H_x \longrightarrow H \longrightarrow H_\Sigma \longrightarrow 0 \tag{49}$$

where H_Σ is the image of H in the group of permutations of Σ. Since the action of H on Σ is transitive, we have that either $H_\Sigma = \mathbb{Z}_4$, or $H_\Sigma = \mathbb{Z}_2^2$. The first possibility $H_\Sigma = \mathbb{Z}_4$, $H_x = \mathbb{Z}_2^3$ is not realized as $\mathfrak{r}(H_x) \leq 2$. Hence we have $H_\Sigma = \mathbb{Z}_2^2$, and either $H_x = \mathbb{Z}_4 \times \mathbb{Z}_2$, or $H_x = \mathbb{Z}_2^3$. Similarly, the second subcase is not realized since $\mathfrak{r}(H_x) \leq 2$. We obtain $H_\Sigma = \mathbb{Z}_2^2$, $H_x = \mathbb{Z}_4 \times \mathbb{Z}_2$. In particular, the exact sequence (49) splits: $H = H_x \times H_\Sigma$.

There exists the following diagram:

$$\begin{array}{ccc} \widetilde{E} \subset \mathrm{Bl}_0\mathbb{C}^2 & \longrightarrow & \mathrm{Bl}_x S \supset E \\ \downarrow & & \downarrow \\ 0 \in \mathbb{C}^2 & \longrightarrow & S \ni x \end{array} \tag{50}$$

where $\mathrm{Bl}_0\mathbb{C}^2$ is the blow up of $\mathbb{C}^2$ at the closed point 0, $\mathrm{Bl}_x S$ is the blow up of S at the closed point x, and horizontal arrows are quotient maps by the action of $\mathbb{Z}_2$. Hence, $\widetilde{H}_x$ faithfully acts on $\mathrm{Bl}_0\mathbb{C}^2$, and H_x faithfully acts on $\mathrm{Bl}_x S$. There are exact sequences:

$$0 \longrightarrow H_n \longrightarrow H_x \longrightarrow H_E \longrightarrow 0 \tag{51}$$

$$0 \to \widetilde{H}_N \to \widetilde{H}_x \to \widetilde{H}_{\widetilde{E}} \to 0, \tag{52}$$

where

1. H_E faithfully acts on the (-2)-curve E on $\mathrm{Bl}_x S$,
2. $\widetilde{H}_{\widetilde{E}}$ faithfully acts on the (-1)-curve $\widetilde{E}$ on $\mathrm{Bl}_0\mathbb{C}^2$,
3. H_N faithfully acts in the normal bundle to E on $\mathrm{Bl}_x S$,
4. $\widetilde{H}_N$ faithfully acts in the normal bundle to $\widetilde{E}$ on $\mathrm{Bl}_0\mathbb{C}^2$.

Since $\widetilde{E}$ is the ramification curve of the quotient map $\mathrm{Bl}_0\mathbb{C}^2 \to \mathrm{Bl}_x S$, we have $H_E = \widetilde{H}_{\widetilde{E}}$, and $\widetilde{H}_N/(\mathbb{Z}_2) = H_N$. From diagram (33) we obtain an exact sequence:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{H}_x \longrightarrow H_x \longrightarrow 0$$

We claim that $\widetilde{H}_x$ is abelian. Indeed, by Proposition 28 we see that $\widetilde{G}_2$ is abelian.

Since $\mathfrak{r}(\widetilde{H}_x)=2$, $\widetilde{H}_x$ is abelian and $H_x=\mathbb{Z}_4\times\mathbb{Z}_2$, we have that either $\widetilde{H}_x=\mathbb{Z}_8\times\mathbb{Z}_2$, or $\widetilde{H}_x=\mathbb{Z}_4^2$.

We show that if the first possibility is realized then G is of product type. Let $\widetilde{H}_x=\mathbb{Z}_8\times\mathbb{Z}_2$. Then $\widetilde{G}_x=\mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_n$ for some $n\geq 1$, according to Remark 31. However, in this case $\mathfrak{r}(G_x)\leq 2$, and hence G is of product type.

Thus, we have $\widetilde{H}_x=\mathbb{Z}_4^2$. Then $\widetilde{H}_{\widetilde{E}}=H_E=\mathbb{Z}_4$ whose generator acts on $\mathbb{C}^2$ via the matrix $\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}$, $\widetilde{H}_N=\mathbb{Z}_4$, whose generator acts on $\mathbb{C}^2$ via the matrix $\begin{pmatrix}\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}$. We have $H_N=\mathbb{Z}_2$. Hence, the exact sequence (51) splits. As shown above, the exact sequence (49) splits as well. Thus, we have

$$H=H_x\times H_\Sigma=H_N\times H_E\times H_\Sigma=\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_2^2.$$

From the local description it follows that $\widetilde{H}_N$ preserves the standard form $dx\wedge dy$. It follows that H_N acts symplectically on S. Hence, there exists a symplectic element of order 4 in $H=\mathbb{Z}_4\times\mathbb{Z}_2^3$. However, this contradicts to Theorem 18. □

5.9 Case $h^0(-K_X)=1$ with cyclic quotient singularities

In this section, we assume that X the non-Gorenstein locus of X consists of cyclic quotient singularities. We assume that it is not only composed of singularities of type $1/2(1,1,1)$, since it has already been treated in Section 5.8. We prove the following

Theorem 53. Let X be a $G\mathbb{Q}$-Fano threefold where G is a finite abelian group. Assume that $h^0(-K_X)=1$. Also, assume that the non-Gorenstein locus of X consists only of cyclic quotient singularities. Then G is of product type.

The next table was obtained from the table in Proposition 35, using the assumption that all the non-Gorenstein points on X are cyclic quotient singularities.

Singularities	**Possibilities for H**	**Argument**
$2\times 1/10(3,7,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$	Lemma 60
$3\times 1/7(3,4,1)$	$\mathbb{Z}_6\times\mathbb{Z}_3^2$	Lemma 54
$2\times 1/11(4,7,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$	Lemma 60
$6\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_6\times\mathbb{Z}_3^2$	Lemma 60, Lemma 54
$2\times 1/9(2,7,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$	Lemma 60
$4\times 1/2(1,1,1)$, $4\times 1/3(1,2,1)$	$\mathbb{Z}_4\times\mathbb{Z}_2^3$	Lemma 61
$4\times 1/5(2,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2^3$	Lemma 60, Lemma 61

$4 \times 1/2(1,1,1), 4 \times 1/4(1,3,1)$	$\mathbb{Z}_4 \times \mathbb{Z}_2^3$	Lemma 61
$2 \times 1/11(3,8,1)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	Lemma 60
$8 \times 1/3(1,2,1)$	$\mathbb{Z}_2^5$, $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2^3$	Lemma 55, Lemma 60, Lemma 61
$3 \times 1/7(2,5,1)$	$\mathbb{Z}_6 \times \mathbb{Z}_3^2$	Lemma 54
$2 \times 1/11(2,9,1)$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	Lemma 60
$8 \times 1/2(1,1,1)$, $4 \times 1/3(1,2,1)$	$\mathbb{Z}_4 \times \mathbb{Z}_2^3$	Lemma 61

Table 7. Possible baskets of cyclic quotient singularities

Lemma 54. If H is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_3^2$, then G is of product type.

Proof. By Proposition 20 we have $\mathrm{rk}(\mathrm{Pic}^H(S')) \leq 2$, leaving us with only the following possibilities among those presented in Proposition 35:

1. $3 \times 1/7(3,4,1)$,
2. $6 \times 1/4(1,3,1)$,
3. $3 \times 1/7(2,5,1)$.

By Corollary 30 we may assume that S has singularity of type A_6, A_3 and A_6, respectively, in each of the three cases. However, it contradicts Corollary 22. □

Lemma 55. If $H = \mathbb{Z}_2^5$, then G is of product type.

Proof. By Proposition 35, the only possible basket of singularities such that G may not be of product type is $8 \times 1/3(1,2,1)$. By Lemma 43, the 8 non-Gorenstein singular points of X belong to one G-orbit (and hence to one H-orbit as well).

By Corollary 30, we may assume that on S these singular points have type A_2. Let x be such a point. We have the following diagram:

$$\begin{array}{ccc} (\widetilde{x} \in \widetilde{X}) \simeq (0 \in \mathbb{C}^3) & \xrightarrow{\pi} & X \ni x \\ \uparrow & & \uparrow \\ (\widetilde{x} \in \widetilde{S}) \simeq (0 \in \mathbb{C}^2) & \xrightarrow{\pi|_{\widetilde{S}}} & S \ni x \end{array} \tag{56}$$

There is an induced diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_3 & \longrightarrow & \widetilde{G}_x & \longrightarrow & G_x & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_3 & \longrightarrow & \widetilde{H}_x & \longrightarrow & H_x & \longrightarrow & 0 \end{array} \tag{57}$$

Now let $\Sigma=\{x=x_1,\dots,x_8\}$ be a G-orbit (or, equivalently, an H-orbit) of x. Consider the following exact sequence

$$0 \longrightarrow H_x \longrightarrow H \longrightarrow H_\Sigma \longrightarrow 0 \tag{58}$$

where H_Σ is the image of H in the group of permutations of Σ. Since the action of H on Σ is transitive, we have $H_\Sigma=\mathbb{Z}_2^3$. We obtain $H_x=\mathbb{Z}_2^2$. In particular, (58) splits: $H=H_x\times H_\Sigma$.

Consider an exact sequence $0 \longrightarrow H_s \longrightarrow H \longrightarrow H_{ns} \longrightarrow 0$ where H_s is a subgroup that acts on the minimal resolution of S symplectically, and H_{ns} is a cyclic group. By Corollary 13, we have $H_s=\mathbb{Z}_2^4$, and $H_{ns}=\mathbb{Z}_2$.

Hence there exists a non-trivial element in H_x that acts symplectically, call it α. In other words, α is a Nikulin involution. We know that it has exactly 8 fixed points, see Remark 10. It follows that α permutes (otherwise it would have more than 8 fixed points) (-2)-curves over each point x_i in the minimal resolution of S. We have the following diagram:

$$\begin{array}{ccc}
\widetilde{E}',\widetilde{E_1},\widetilde{E_2}\subset\widetilde{\mathrm{Bl}_0\mathbb{C}^2} & \xrightarrow{3:1} & \widetilde{\mathrm{Bl}_x S}\supset E,E_1',E_2' \\
\downarrow & & \downarrow \\
\widetilde{E}\subset \mathrm{Bl}_0\mathbb{C}^2 & & \mathrm{Bl}_x S\supset E_1,E_2 \\
\downarrow & & \downarrow \\
0\in\mathbb{C}^2 & \xrightarrow{3:1} & S\ni x
\end{array} \tag{59}$$

where

1. $\mathrm{Bl}_0\mathbb{C}^2$ is the blow up of $\mathbb{C}^2$ at the closed point 0 with the exceptional (-2)-curves E_1 and E_2,
2. $\mathrm{Bl}_x S$ is the blow up of S at the closed point x with the exceptional (-1)-curve $\widetilde{E}$,
3. $\widetilde{\mathrm{Bl}_0\mathbb{C}^2}$ is the blow up of two $\mathbb{Z}_3$-fixed points on $\widetilde{E}$, where $\widetilde{E_1}$ and $\widetilde{E_2}$ are (-1)-curves, and $\widetilde{E}'$ is a smooth rational (-3)-curve,
4. $\widetilde{\mathrm{Bl}_x S}$ is the blow up of the intersection point of E_1 with E_2, so E' is a (-1)-curve, E_1' and E_2' are smooth rational (-3)-curves.

Since α is symplectic, it lifts to an element that acts on $\mathbb{C}^2$ via the matrix $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. However, in this case the lift of α to $\widetilde{\mathrm{Bl}_0\mathbb{C}^2}$ does not interchange $\widetilde{E_1}$ and $\widetilde{E_2}$, hence it does not interchange E_1 with E_2. This is a contradiction. □

Lemma 60. If H is isomorphic to $\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$, then G is of product type.

Proof. The only possibilities for non-Gorenstein points of X are the following.

1. $2\times 1/10(3,7,1)$,

2. $2 \times 1/11(4,7,1)$,
3. $6 \times 1/4(1,3,1)$,
4. $2 \times 1/9(2,7,1)$,
5. $4 \times 1/5(2,3,1)$,
6. $2 \times 1/11(3,8,1)$,
7. $8 \times 1/3(1,2,1)$,
8. $2 \times 1/11(2,9,1)$.

By Corollary 30, we may assume that S has k singular points of type A_{r_i-1} that correspond to k quotient singularities of index r_i in the list above. However, by the Corollary 22 we see that r_i could be equal only to 2 or 3. This leaves us with the only case $8 \times 1/3(1,2,1)$. Thus S has 8 singular points of type A_2.

According to Corollary 13, we have $H = H_s \times H_{ns}$, where $H_s = \mathbb{Z}_4 \times \mathbb{Z}_2$, $H_{ns} = \mathbb{Z}_8$. By Proposition 28 we know that since $r = 3$, the lifting $\widetilde{G}_x$ is abelian. Hence, the lifting $\widetilde{H}_x$ is abelian as well. Thus $\widetilde{H}_x$ does not interchange the two (-2)-curves E_1 and E_2 as in diagram (59). However, this contradicts to Corollary 22 as there are at least 2 H-orbits of (-2)-curves on the minimal resolution of S. □

Lemma 61. If H is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2^3$, then G is of product type.

Proof. By Proposition 35, the possibilities for the basket of singularities of X are the following.

1. $4 \times 1/2(1,1,1), 4 \times 1/3(1,2,1)$
2. $4 \times 1/5(2,3,1)$
3. $4 \times 1/2(1,1,1), 4 \times 1/4(1,3,1)$
4. $8 \times 1/3(1,2,1)$
5. $8 \times 1/2(1,1,1), 4 \times 1/3(1,2,1)$

Using Proposition 38, we exclude the cases (1) and (3) which leaves us with

1. $4 \times 1/5(2,3,1)$
2. $8 \times 1/3(1,2,1)$
3. $8 \times 1/2(1,1,1), 4 \times 1/3(1,2,1)$

Consider the case $8 \times 1/3(1,2,1)$. Using (Loginov, 2024, Proposition 7.13) we know that since $r = 3$, the lifting $\widetilde{G}_x$ is abelian. Hence, the lifting $\widetilde{H}_x$ is abelian as well. Thus it does not interchange the two (-2)-curves E_1 and E_2 as in diagram (59).

Let $\Sigma = \{x = x_1, \ldots, x_8\}$ be a G-orbit (or, equivalently, an H-orbit) of x. Consider the exact sequence

$$0 \longrightarrow H_x \longrightarrow H \longrightarrow H_\Sigma \longrightarrow 0 \qquad (62)$$

where H_Σ is the image of H in the group of permutations of Σ.

Consider the case when (49) splits: $H = H_x \times H_\Sigma$, and $H_\Sigma = \mathbb{Z}_2^3$, $H_x = \mathbb{Z}_4$.

We may assume that $H_s = \mathbb{Z}_2^3$, and $H_{ns} = \mathbb{Z}_4$ (otherwise H_x contains a Nikulin involution which has exactly 8 fixed points, so E_1 and E_2 should be permuted which contradicts to the fact that the lifting $\widetilde{G}_x$ is abelian).

The local computation shows that σ has 8 fixed (-2)-curves $E_1, \ldots, E_8$ and there are 8 (-2)-curves $E_1', \ldots, E_8'$ which are preserved by σ, so that $E_i \cdot E_i' = 1$, $E_i \cdot E_j' = 0$ for $i \neq j$.

Also, one checks that there exists a curve C which is preserved by σ so that $C \cdot E_i' = 1$, $C \cdot E_i = 0$ for any i, and σ^2 fixes C pointwise. Observe that $\mathbb{Z}_2^4$ acts on C faithfully.

Assume that C is reducible. Then by Zhang (1998) it has two components $C = C_1 + C_2$ so that C_1 and C_2 are permuted by $\mathbb{Z}_2^4$. Hence $\mathbb{Z}_2^3$ acts on each component faithfully. Thus C_i are not rational curves. We arrive at a contradiction with (Zhang, 1998, Theorem 3).

Assume that C is irreducible. Then it is easy to check that $g(C) \geq 3$. However, this contradicts to the classification of the fixed locus of non-symplectic involutions, see (Brandhorst & Hofmann, 2023, Theorem 1.4).

Consider the case $8 \times 1/2(1,1,1), 4 \times 1/3(1,2,1)$. Consider the orbit $4 \times 1/3(1,2,1)$. The stabilizer of a point should contain a Nikulin involution. Since the lifting $\widetilde{H}_x$ is abelian, it follows that two (-2)-curve over x are not interchanged. Hence the Nikulin involution stabilizes 8 smooth rational curves, so it has more than 8 fixed points, which is a contradiction.

Finally, consider the case $4 \times 1/5(2,3,1)$. The same argument as in the previous case applies. □

5.9.1 Index 2 case

We treat the case when X is a $G\mathbb{Q}$-Fano threefold of index at least 2. It turns out that the index of X equal to 2, and the following cases are possible:

1. $2 \times 1/3(1,2,2), 2 \times 1/7(3,4,2)$,
2. $4 \times 1/3(1,2,2), 2 \times 1/5(1,4,2)$,
3. $2 \times 1/5(2,3,2), 2 \times 1/7(1,6,2)$,
4. $2 \times 1/11(4,7,2)$,
5. $2 \times 1/5(1,4,2), 2 \times 1/7(3,4,2)$,
6. $3 \times 1/3(1,2,2), 3 \times 1/5(1,4,2)$,
7. $3 \times 1/7(3,4,2)$,
8. $2 \times 1/3(1,2,2), 2 \times 1/9(4,5,2)$.

Proposition 39. The index 2 case is not realized.

Proof. Using Corollary 30, we may assume that an index r cyclic quotient singularity $x \in X$ corresponds to a singular point $x \in S$ of type A_{r-1} on S. Note that in each case we have singularities A_n on S for $n > 2$. On the other hand, the group H can be isomorphic either to $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, or to $\mathbb{Z}_6 \times \mathbb{Z}_3^2$. But then we get a contradiction with Corollary 22. □

5.10 Case $h^0(-K_X) = 1$ with terminal singularities

In this section, we prove the following.

Theorem 63. Let X be a $G\mathbb{Q}$-Fano threefold where G is a finite abelian group. Assume that $h^0(-K_X) = 1$. Then G is of product type.

Proposition 40. Assume that all the non-Gorenstein points on X have index 2. Then G is of product type.

Proof. If all of them are cyclic quotient singularities, then this case is already considered in Section 5.8. Hence we may assume that at least one point, say $P_1 \in X$, is not a cyclic quotient singularity. Consider its basket $\{P_{1,j}\}$, $1 \leq j \leq b_1$. By the discussion in 5.5.1 we have $b_1 \geq 2$.

By Lemma 43 and Lemma 44 we see that the groups $H = \mathbb{Z}_2^5$ and $H = \mathbb{Z}_4 \times \mathbb{Z}_2^3$ are excluded, since in this case k_1 is divisible by 4. Hence the total number N of half-points in the baskets of $k_1 \times P_1$ is divisible by 8. However, by Proposition 36 we know that $9 \leq N \leq 15$. This is a contradiction.

Consider the case $H = \mathbb{Z}_6 \times \mathbb{Z}_3^2$. By Lemma 42 we have that k_1 is divisible by 3. Hence the total number N of half-points in the baskets of $k_1 \times P_1$ is divisible by 6. However, by Proposition 36 we know that $9 \leq N \leq 15$. Thus $N = 12$, and so $k_1 = 6$. By Lemma 42 we see that X has no other singular points. Also, S has only singular points of type A_1 or A_2. However, the second case is impossible as H acts on the singular points $x_1, \ldots, x_6$ transitively. We conclude that S has 6 points of type A_1. Consider the index 1 cover $\widetilde{x} \in \widetilde{X} \to X \ni x$ and the induced cover $\widetilde{x} \in \widetilde{S} \to S \ni x$. Note that $\widetilde{S}$ is smooth since $x \in S$ is a A_1 singularity. However, $\widetilde{x} \in \widetilde{X}$ is a Gorenstein singular point, and $\widetilde{S}$ is a Cartier divisor at $\widetilde{x} \in \widetilde{X}$. Thus $\widetilde{S}$ should be singular at $\widetilde{x}$, which is a contradiction.

Consider the case $H = \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$. By Lemma 41 we have that k_1 is divisible by 2. Hence the total number N of half-points in the baskets of $k_1 \times P_1$ is divisible by 4. However, by Proposition 36 we know that $9 \leq N \leq 15$. Thus $N = 12$, and so $k_1 = 6$. However, in this case G has at least two orbits of singular points, which contradicts Lemma 41. □

We have the following list of remaining possibilities.

Singularities	Basket of X	Possibilities for H
$2\times cA/4$ or $3\times cA/4$	$6\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_6\times\mathbb{Z}_3^2$
$2\times cA/4$	$4\times 1/5(2,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$4\times cA/3$ or $4\times cD/3$	$8\times 1/3(1,2,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2^3$
$2\times cAx/4$	$8\times 1/2(1,1,1)$, $2\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$2\times cAx/4$	$10\times 1/2(1,1,1)$, $2\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$4\times cA/2+4\times 1/3(1,2,1)$	$8\times 1/2(1,1,1)$, $4\times 1/3(1,2,1)$	$\mathbb{Z}_4\times\mathbb{Z}_2^3$
$2\times cAx/4$	$6\times 1/2(1,1,1)$,$2\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$4\times cAx/4$	$4\times 1/2(1,1,1)$,$4\times 1/4(1,3,1)$	$\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2$
$3\times cAx/4$	$6\times 1/2(1,1,1)$,$3\times 1/4(1,3,1)$	$\mathbb{Z}_6\times\mathbb{Z}_3^2$

Table 8. Possible baskets of singularities on X

Proposition 32 together with Corollary 22 exclude all cases except for the following:

1. $4\times cA/3$, $H=\mathbb{Z}_4\times\mathbb{Z}_2^3$,
2. $4\times cD/3$, $H=\mathbb{Z}_4\times\mathbb{Z}_2^3$,
3. $4\times cA/2+4\times 1/3(1,2,1)$, $H=\mathbb{Z}_4\times\mathbb{Z}_2^3$.

In the case (3), by Corollary 30 and Proposition 32 we may assume that we have $4A_2$ singularities on S that correspond to $4\times 1/3(1,1,1)$, and 4 du Val singularities of type different from A_1 or A_2. Denote by $f\colon S'\to S$ the minimal resolution of S. It follows that there are at least 20 f-exceptional (-2)-curves. However, this contradicts to the fact that on a smooth K3 surface S' one has $\rho(S')\leq 20$. Hence this case is not realized.

Consider the cases (1) (or (2)), that is, when X has singularities $4\times cA/3$ (or $4\times cD/3$), and $H=\mathbb{Z}_4\times\mathbb{Z}_2^3$. By Proposition 32 and Remark 26, we see that S has 4 singularities of type A_k where $k\geq 5$. However, arguing as in the previous case we get a contradiction with $\rho(S')\leq 20$. Hence this case is not realized as well. This proves Theorem 63.

5.11 Proof of main results

Proof of Theorem 4. Assume that G is a group that faithfully acts on a rational connected threefold X. By a standard argument, we may assume that X is a projective $G\mathbb{Q}$-Mori fiber space over the base Z. If $\dim Z>0$ then G is of product type by (Loginov, 2024, Corollary 3.17). Hence we may assume that X is a $G\mathbb{Q}$-Fano threefold.

If $h^0(-K_X)=0$ and for any $G\mathbb{Q}$-Fano threefold X' which is G-birational to X we have $h^0(-K_{X'})=0$, then G is of type (3) as in Theorem 4. Hence we may assume that $h^0(-K_X)>0$. Also, by Loginov (2024) we may assume that for any G-invariant element $S\in|-K_X|$, the pair (X,S) is plt, so S is a K3 surface with at worst du Val singularities. If $h^0(-K_X)\geq 2$ then by Theorem 37 we have that G is of product type. So we may assume that $h^0(-K_X)=1$. In particular, this implies that the set of non-Gorenstein singularities of X is non-empty.

If the set of non-Gorenstein singularities consists of points of type $1/2(1,1,1)$, then by Theorem 45 we have that G is of product type. If the set of non-Gorenstein singularities consists of cyclic quotient singularities, then by Theorem 53 we have that G is of product type. Finally, if the set of non-Gorenstein singularities consists of terminal points whic hare non necessarily cyclic quotient singularities, then by Theorem 63 we have that G is of product type. □

5.12 Appendix: Intersection matrices for K3 surfaces acted on by $\mathbb{Z}_2^5$ and $\mathbb{Z}_4\times\mathbb{Z}_2^3$

Proposition 41. Let S be a $K3$ surface with a faithful action of $H=\mathbb{Z}_4\times\mathbb{Z}_2^3$. Then the intersection matrix M on $\mathrm{Pic}^H(S)$ is one of the following.

1.
$$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix},$$

2.
$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

3.
$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & -8 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

4.
$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix},$$

5.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & -4 & -2 & 2 & 2 & 0 \\ 0 & -2 & -4 & 2 & 2 & 0 \\ 0 & 2 & 2 & -4 & 0 & 0 \\ 0 & 2 & 2 & 0 & -4 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

6.

$$\begin{pmatrix} 0 & -2 & -2 & 2 & -2 & 0 \\ -2 & -4 & -2 & 0 & -2 & -2 \\ -2 & -2 & -4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & -4 & 0 \\ 0 & -2 & 0 & 0 & 0 & -4 \end{pmatrix},$$

7.

$$\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -4 & -2 & 0 & -2 & -2 \\ 0 & -2 & -4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -4 & 0 \\ 0 & -2 & 0 & 0 & 0 & -4 \end{pmatrix},$$

8.

$$\begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & -2 & -6 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -4 & -6 & 2 \\ 0 & -6 & 0 & -6 & -20 & 8 \\ 0 & 2 & 0 & 2 & 8 & -4 \end{pmatrix},$$

9.

$$\begin{pmatrix} 0 & -2 & 2 & -4 & -4 & 2 \\ -2 & -4 & 0 & -4 & -6 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ -4 & -4 & 0 & -8 & -8 & 4 \\ -4 & -6 & 0 & -8 & -12 & 4 \\ 2 & 2 & 0 & 4 & 4 & -4 \end{pmatrix},$$

10.

$$\begin{pmatrix} 0 & 0 & -2 & -4 & -2 & -6 \\ 0 & -8 & -2 & -20 & -14 & -26 \\ -2 & -2 & -4 & -8 & -4 & -12 \\ -4 & -20 & -8 & -60 & -40 & -78 \\ -2 & -14 & -4 & -40 & -28 & -52 \\ -6 & -26 & -12 & -78 & -52 & -104 \end{pmatrix}.$$

Proposition 42. Let S be a $K3$ surface with a faithful action of $H = \mathbb{Z}_2^5$. Then the intersection matrix M on $\mathrm{Pic}^H(S)$ is one of the following. We will also denote by A the restriction of M to its anisotropic part in the Witt decomposition of M over $\mathbb{Z}$. Moreover, the anisotropic part we present is not further decomposable into orthogonal sublattices.

1.

$$\begin{pmatrix} 8 \end{pmatrix},$$

2.

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

3.

$$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix},$$

4.

$$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix},$$

5.

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & -8 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

6.

$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix},$$

7.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & -4 & -2 & -2 & 0 \\ 0 & -2 & -4 & -2 & 0 \\ 0 & -2 & -2 & -4 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

8.

$$\begin{pmatrix} -4 & -4 & 0 & -2 & -2 \\ -4 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ -2 & 0 & -2 & -4 & 0 \\ -2 & 0 & 0 & 0 & -4 \end{pmatrix},$$

9.

$$\begin{pmatrix} -4 & 2 & -4 & -6 & 2 \\ 2 & 0 & 0 & 0 & 0 \\ -4 & 0 & -4 & -2 & 0 \\ -6 & 0 & -2 & -12 & 6 \\ 2 & 0 & 0 & 6 & -4 \end{pmatrix},$$

10.

$$\begin{pmatrix} -52 & -10 & -20 & -38 & -28 \\ -10 & -4 & -4 & -6 & -4 \\ -20 & -4 & -8 & -14 & -10 \\ -38 & -6 & -14 & -28 & -20 \\ -28 & -4 & -10 & -20 & -16 \end{pmatrix},$$

11.

$$\begin{pmatrix} -4 & 0 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ -2 & 0 & -4 & 0 & 0 \\ -4 & 0 & 0 & -8 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix},$$

12.

$$\begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & -4 & 0 & -4 & -2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & -12 & -6 \\ 0 & -2 & 0 & -6 & -4 \end{pmatrix},$$

13.

$$\begin{pmatrix} 0 & 2 & 2 & -2 & 0 \\ 2 & 4 & 4 & -2 & -2 \\ 2 & 4 & 0 & 0 & 0 \\ -2 & -2 & 0 & -4 & 0 \\ 0 & -2 & 0 & 0 & -4 \end{pmatrix},$$

14.

$$\begin{pmatrix} -4 & -4 & -2 & -2 & 0 \\ -4 & -8 & 0 & 0 & 0 \\ -2 & 0 & -4 & 0 & 0 \\ -2 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 2 & -4 \end{pmatrix}.$$

Chapter 6

Groups acting on Fano threefolds in the family №2.12, and K-stability

"No day without a line."

Natalia Cheltsova

In this chapter, we present the results obtained in Cheltsov, Li, et al. (2024). We describe all groups that can act faithfully on Fano threefolds that are the smooth complete intersection of three divisors of bidegree $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. These threefolds are rational since they are isomorphic to the blow-up of $\mathbb{P}^3$ along a smooth curve of degree six and genus three. They are particularly attractive to us since their automorphism groups are either isomorphic to groups of automorphisms of smooth plane quartics, or a double extension of them. The reader might notice that we did not give this chapter the name of the article it presents. This is because this thesis focuses on groups of symmetries, whereas the paper was written with the aim of describing the K-stability of those threefolds. It is also the reason why we end this Ph.D. thesis with this work; we will make an opening to the world of K-stability. Recall one of the most celebrated results in modern geometry, called the Yau-Tian-Donaldson conjecture and proven in Chen, Donaldson, and Sun (2015).

Theorem 43. A smooth Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable.

We will present many examples of K-polystable Fano threefolds in the family described above. All authors have approved the inclusion of this work in the present thesis and acknowledge equal contribution.

6.1 Introduction

Let C be a smooth quartic curve in $\mathbb{P}^2$, let D be a divisor of degree 2 on the curve C such that

$$h^0\big(\mathscr{O}_C(D)\big) = 0. \tag{1}$$

Then $K_C + D$ is very ample, see Homma (1980), and the linear system $|K_C + D|$ gives an embedding $\varphi\colon C \hookrightarrow \mathbb{P}^3$. We set $C_6 = \varphi(C)$. Then C_6 is a smooth curve of degree 6 and genus 3.

Let $\pi\colon X \to \mathbb{P}^3$ be the blow up of the curve C_6. Then X is a Fano threefold in the deformation family №2.12 in the Mori–Mukai list, and every smooth member of this family can be obtained in this way. Moreover, the Fano threefold X can be given in $\mathbb{P}^3 \times \mathbb{P}^3$ by

$$(x_0, x_1, x_2, x_3)M_1 \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = (x_0, x_1, x_2, x_3)M_2 \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = (x_0, x_1, x_2, x_3)M_3 \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0 \tag{2}$$

for appropriate 4×4 matrices M_1, M_2, M_3 such that π is induced by the projection to the first factor, where $([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3])$ are coordinates on $\mathbb{P}^3 \times \mathbb{P}^3$.

Let $\pi'\colon X \to \mathbb{P}^3$ be the morphism induced by the projection $\mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ to the second factor. Then π' is a blow up of $\mathbb{P}^3$ along a smooth curve C_6' of degree 6 and genus 3, and the π'-exceptional surface is spanned by the strict transforms of the trisecants of the curve C_6. Furthermore, we have the following commutative diagram:

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\pi'} \\ \mathbb{P}^3 & \overset{\chi}{\dashrightarrow} & \mathbb{P}^3 \end{array} \tag{3}$$

where χ is the birational map given by the linear system consisting of all cubic surfaces containing C_6. Note that the curves C_6 and C_6' are isomorphic, but not necessarily projectively isomorphic.

We can find the equations of the curves C_6 and C_6' as follows. Rewrite (2) as

$$\begin{cases} L_{10}y_0 + L_{11}y_1 + L_{12}y_2 + L_{13}y_3 = 0, \\ L_{20}y_0 + L_{21}y_1 + L_{22}y_2 + L_{23}y_3 = 0, \\ L_{30}y_0 + L_{31}y_1 + L_{32}y_2 + L_{33}y_3 = 0, \end{cases}$$

where the L_{ij}'s are linear functions in x_0, x_1, x_2, x_3. Set

$$M = \begin{pmatrix} L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix}.$$

Let f_0, f_1, f_2, f_3 be the determinants of the 3×3 matrices obtained from the matrix M by removing its first, second, third, fourth columns, respectively. Then $C_6 = \{f_0 = 0, f_1 = 0, f_2 = 0, f_3 = 0\}$, and the birational map $\chi \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ in the diagram (3) is given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto \big[f_0 : f_1 : f_2 : f_3\big]$$

up to a composition with an automorphism of the projective space $\mathbb{P}^3$. Similarly, one can also describe the defining equations of the sextic curve C_6'.

Example 44 ((Araujo et al., 2023b, Section 5.4),(W. L. Edge, 1947, (3.4)))**.** Let

$$X = \Big\{x_0y_1 + x_1y_0 - \sqrt{2}x_2y_2 = 0, x_0y_2 + x_2y_0 - \sqrt{2}x_3y_3 = 0, x_0y_3 + x_3y_0 - \sqrt{2}x_1y_1 = 0\Big\} \subset \mathbb{P}^3 \times \mathbb{P}^3.$$

Then X is a smooth Fano threefold in the family №2.12, the curve C_6 is given by

$$\begin{cases} 2\sqrt{2}x_1x_2x_3 - x_0^3 = 0, \\ x_0^2x_1 + \sqrt{2}x_0x_2^2 + 2x_2x_3^2 = 0, \\ x_0^2x_2 + \sqrt{2}x_0x_3^2 + 2x_1^2x_3 = 0, \\ x_0^2x_3 + \sqrt{2}x_0x_1^2 + 2x_1x_2^2 = 0, \end{cases}$$

and C_6' is given by the same equations replacing each x_i by y_i. One has $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}_2$, and X is the only smooth Fano threefold in the deformation family №2.12 that admits a faithful action of $\mathrm{PSL}_2(\mathbb{F}_7)$ (see Corollary 6.4.12). The map χ in (3) can be chosen to be an involution.

The following result has been proven in Araujo et al. (2023b).

Theorem 6.1.1 ((Araujo et al., 2023b, § 5.4))**.** *Let X be the Fano threefold from Example 44. Then X is K-stable.*

Hence, a general member of the family №2.12 is K-stable, since K-stability is an open condition. We expect that every smooth Fano threefold in this family is K-stable. To show this, it is enough to prove that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) > 0$$

for every prime divisor $\mathbf{F}$ over X, see Fujita (2019); Li (2017), where $A_X(\mathbf{F})$ is the log discrepancy of the divisor $\mathbf{F}$, and

$$S_X\big(\mathbf{F}\big)=\frac{1}{(-K_X)^3}\int\limits_0^\infty \mathrm{vol}\big(-K_X-u\mathbf{F}\big)du.$$

Unfortunately, we are unable to prove this result at the moment. Instead, we prove a weaker result. To state it, let E be the π-exceptional surface, and let E' be the π'-exceptional surface.

Theorem A. Let $\mathbf{F}$ be a prime divisor over X such that $\beta(\mathbf{F})\leqslant 0$, and let Z be its center on X. Then Z is a point in the intersection $E\cap E'$.

Let us present applications of this result. Since $\mathrm{Aut}(X)$ is finite, see Cheltsov, Przyjalkowski, and Shramov (2018), the threefold X is K-polystable if and only if it is K-stable. Thus, by (Zhuang, 2021, Corollary 4.14) and (Araujo et al., 2023b, Corollary 1.1.6), Theorem A implies

Corollary 6.1.2. *If $\mathrm{Aut}(X)$ does not fix any point in $E\cap E'$, then X is K-stable.*

Since the action of the group $\mathrm{Aut}(\mathbb{P}^3,C_6)$ lifts to X, Corollary 6.1.2 implies

Corollary 6.1.3. *If $\mathrm{Aut}(\mathbb{P}^3,C_6)$ does not fix a point in C_6, then X is K-stable.*

Since the group $\mathrm{Aut}(\mathbb{P}^3,C_6)$ acts faithfully on the curve C_6, Corollary 6.1.3 implies the following generalization of Theorem 6.1.1, which has more applications (see Section 6.2).

Corollary 6.1.4. *If $\mathrm{Aut}(\mathbb{P}^3,C_6)$ is not cyclic, then X is K-stable.*

Proof. If the group $\mathrm{Aut}(\mathbb{P}^3,C_6)$ fixes a point $P\in C_6$, it acts faithfully on the one-dimensional tangent space to the curve C_6 at the point P by (Flenner & Zaidenberg, 2005, Lemma 2.7), so that $\mathrm{Aut}(\mathbb{P}^3,C_6)$ is cyclic. □

What do we know about $\mathrm{Aut}(X)$? As we already mentioned above, this group is finite, by Cheltsov et al. (2018), and we have the following exact sequence:

$$1\to\mathrm{Aut}\big(\mathbb{P}^3,C_6\big)\to\mathrm{Aut}(X)\to\mathbb{Z}_2,$$

where $\mathrm{Aut}(\mathbb{P}^3,C_6)\simeq\mathrm{Aut}(C,[D])$, and the final homomorphism is surjective if and only if $\mathrm{Aut}(X)$ has an element that swaps E and E'. For instance, if X is the smooth Fano threefold from Example 44, then the group $\mathrm{Aut}(X)$ contains such an element — it is the involution given by

$$\big([x_0:x_1:x_2:x_3],[y_0:y_1:y_2:y_3]\big)\mapsto\big([y_0:y_1:y_2:y_3],[x_0:x_1:x_2:x_3]\big),$$

which implies that $\mathrm{Aut}(X)\simeq\mathrm{PSL}_2(\mathbb{F}_7)\times\mathbb{Z}_2$ in this case. In Section 6.4, we will discuss the possibilities for the group $\mathrm{Aut}(X)$ in more detail. In particular, we will present a criterion when $\mathrm{Aut}(X)$ contains an element that swaps E and E', and we will prove the following result (cf. (Wei & Yu, 2020, Theorem 1.1)).

Theorem B. A finite group G has a faithful action on a smooth Fano threefold in the deformation family №2.12 if and only if G is isomorphic to a subgroup of $\mathrm{PSL}_2(\mathbb{F}_7)\times\mathbb{Z}_2$ or $\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$.

As we mentioned in Example 44, the family №2.12 contains a unique smooth Fano threefold that admits a faithful action of the group $\mathrm{PSL}_2(\mathbb{F}_7)$. Similarly, we prove in Section 6.4 that the deformation family №2.12 contains a unique smooth threefold that admits a faithful action of the group $\mathbb{Z}_4^2\rtimes\mathbb{Z}_3$, and the full automorphism group of this threefold is $\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$.

Remark 6.1.5. Let G be a subgroup in $\mathrm{Aut}(X)$. If G has an element that swaps the surfaces E and E', then X is a G-Mori fiber space (over a point), and X is also known as a G-Fano threefold (see Y. Prokhorov (2013)). In this case, it is natural to ask the following three nested questions:

1. Is there a G-equivariant birational map $X\dashrightarrow\mathbb{P}^3$? cf. Cheltsov, Tschinkel, and Zhang (2024); Ciurca et al. (2024); Kuznetsov and Prokhorov (2021).
2. Is X G-solid? cf. Cheltsov and Sarikyan (2022); Pinardin (2024).
3. Is X G-birationally rigid? cf. Cheltsov and Shramov (2016a).

Inspired by (Kuznetsov & Prokhorov, 2021, Corollary 6.11), we conjecture that the answer to the first question is always negative. If $G\simeq\mathrm{PSL}_2(\mathbb{F}_7)\times\mathbb{Z}_2$, then X is G-birationally rigid by (Araujo et al., 2023b, Theorem 5.23), so, in particular, it is G-solid. We believe that X is also G-birationally rigid if $G\simeq\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$.

To consider more applications of Theorem A, let $\mathbb{k}$ be a subfield in $\mathbb{C}$ such that C_6 is defined over $\mathbb{k}$. Then X and the Sarkisov link (3) are defined over $\mathbb{k}$. In particular, the curve C_6' is defined over $\mathbb{k}$. Moreover, it follows from Benoist and Wittenberg (2019); Lauter (2001) that C_6' and C_6 are isomorphic over $\mathbb{k}$, which can be shown directly. By (Zhuang, 2021, Corollary 4.14), Theorem A implies the following corollaries.

Corollary 6.1.6. *If $E\cap E'$ does not have $\mathbb{k}$-points, then X is K-stable.*

Corollary 6.1.7. *If C_6 does not have $\mathbb{k}$-points, then X is K-stable.*

Using (Zhuang, 2021, Corollary 4.14), we also obtain the following.

Corollary 6.1.8. *Every smooth Fano threefold in the deformation family №2.12 which is defined over a subfield of the field $\mathbb{C}$ and does not have points in this subfield is K-stable.*

We will present applications of Corollaries 6.1.7 and 6.1.8 in Section 6.2.

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6.2 Examples

6.2.1 $\mathfrak{S}_4$-invariant curves

Let

$$M_1=\begin{pmatrix}0&a&1&0\\a&0&0&-1\\1&0&0&a\\0&-1&a&0\end{pmatrix},M_2=\begin{pmatrix}0&1&0&a\\1&0&a&0\\0&a&0&-1\\a&0&-1&0\end{pmatrix},M_3=\begin{pmatrix}0&0&a&-1\\0&0&1&a\\a&1&0&0\\-1&a&0&0\end{pmatrix},$$

where $a\in\mathbb{C}$ satisfies $a(a^6-1)(a^2+1)\neq 0$. Consider the plane quartic curve $C\subseteq\mathbb{P}^2_{x,y,z}$ defined by

$$\det(xM_1+yM_2+zM_3)=x^4+y^4+z^4+\lambda(x^2y^2+x^2z^2+y^2z^2)=0, \tag{4}$$

where $\lambda=-\frac{2a^4+2}{(a^2+1)^2}$, cf. (W. Edge, 1938, § 14). One easily checks that C is smooth.

Now let X be the complete intersection in $\mathbb{P}^3\times\mathbb{P}^3$ given by:

$$(x_0,x_1,x_2,x_3)M_1\begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix}=(x_0,x_1,x_2,x_3)M_2\begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix}=(x_0,x_1,x_2,x_3)M_3\begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix}=0 \tag{5}$$

We claim that X is a smooth Fano threefold in the deformation family №2.12. Indeed, reverting to the notation in Section 6.1, set

$$M=\begin{pmatrix}ax_1+x_2&ax_0-x_3&ax_3+x_0&ax_2-x_1\\ax_3+x_1&ax_2+x_0&ax_1-x_3&ax_0-x_2\\ax_2-x_3&ax_3+x_2&ax_0+x_1&ax_1-x_0\end{pmatrix},$$

and set $C_6=\{f_0=0,f_1=0,f_2=0,f_3=0\}$ for

$$\begin{aligned}f_0={}&(1-a^3)x_0^3-(2a^2+2a)x_0^2x_1+(2a^2+2a)x_0^2x_2+\\&+(2a^2+2a)x_0^2x_3+(a^3-1)x_0x_1^2-(2a^2-2a)x_0x_1x_2-(2a^2-2a)x_0x_1x_3+\\&+(a^3-1)x_0x_2^2+(2a^2-2a)x_0x_2x_3+(a^3-1)x_0x_3^2-(2a^3+2)x_1x_2x_3,\end{aligned}$$

$$\begin{aligned}f_1={}&(1-a^3)x_0^2x_1+(-2a^2-2a)x_0x_1^2+(2a^2-2a)x_0x_1x_2-\\&-(2a^2-2a)x_0x_1x_3+(2a^3+2)x_0x_2x_3+(a^3-1)x_1^3+(2a^2+2a)x_1^2x_2-\\&-(2a^2+2a)x_1^2x_3+(-a^3+1)x_1x_2^2+(2a^2-2a)x_1x_2x_3+(1-a^3)x_1x_3^2,\end{aligned}$$

$$
\begin{aligned}
f_2 = {} & (a^3-1)x_0^2x_2-(2a^2-2a)x_0x_1x_2-(2a^3+2)x_0x_1x_3+\\
& +(-2a^2-2a)x_0x_2^2-(2a^2-2a)x_0x_2x_3+(a^3-1)x_1^2x_2+(2a^2+2a)x_1x_2^2+\\
& +(2a^2-2a)x_1x_2x_3+(1-a^3)x_2^3+(2a^2+2a)x_2^2x_3+(a^3-1)x_2x_3^2,
\end{aligned}
$$

$$
\begin{aligned}
f_3 = {} & (1-a^3)x_0^2x_3+(2a^3+2)x_0x_1x_2-(2a^2-2a)x_0x_1x_3-\\
& -(2a^2-2a)x_0x_2x_3+(2a^2+2a)x_0x_3^2+(1-a^3)x_1^2x_3-(2a^2-2a)x_1x_2x_3+\\
& +(2a^2+2a)x_1x_3^2+(1-a^3)x_2^2x_3+(2a^2+2a)x_2x_3^2+(a^3-1)x_3^3.
\end{aligned}
$$

Then it follows from (Ottaviani, 2024, §4.2) that C_6 is isomorphic to the smooth plane quartic C, which implies the claim. Moreover, it follows from Dolgachev (2012) that

$$
\mathrm{Aut}(C_6) \simeq \begin{cases} \mathfrak{S}_4 \text{ if } \lambda \neq 0 \text{ and } \lambda^2+3\lambda+18 \neq 0, \\ \mathbb{Z}_4^2 \rtimes \mathfrak{S}_3 \text{ if } \lambda = 0, \\ \mathrm{PSL}_2(\mathbb{F}_7) \text{ if } \lambda^2+3\lambda+18. \end{cases}
$$

If $\lambda^2+3\lambda+18=0$, then C_6 is isomorphic to the Klein quartic curve.

Lemma 6.2.1. *The group* $\mathrm{Aut}(\mathbb{P}^3, C_6)$ *contains a subgroup isomorphic to* $\mathfrak{S}_4$.

Proof. Let G be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ that is generated by the following transformations:

$$
\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 & -1 & 1 \\ -3 & -1 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} -3 & -1 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ 1 & -3 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}.
$$

Then, using the same Magma code used in Cheltsov and Sarikyan (2022), we see that $G \simeq \mathfrak{S}_4$. Now G, acting on the matrix

$$
A = xM_1 + yM_2 + zM_3
$$

via $g \cdot A = gAg^T$, preserves the quartic (4), thus the curve C_6 is G-invariant (see also (Ottaviani, 2024, §4.2)). □

Corollary 6.2.2. *If* $\lambda \neq 0$ *and* $\lambda^2+3\lambda+18 \neq 0$, *then* $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathfrak{S}_4$.

Similarly, we prove

Lemma 6.2.3. *The group* $\mathrm{Aut}(X)$ *contains a subgroup isomorphic to* $\mathfrak{S}_4 \times \mathbb{Z}_2$.

Proof. Let G be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ that is defined in the proof of Lemma 6.2.1. Then $G \simeq \mathfrak{S}_4$, the group G acts diagonally on $\mathbb{P}^3 \times \mathbb{P}^3$, and X is G-invariant. This gives an embedding $\mathfrak{S}_4 \hookrightarrow \mathrm{Aut}(X)$. Moreover, since the matrices M_1, M_2, M_3 are symmetric, the involution

$$\big([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]\big) \mapsto \big([y_0 : y_1 : y_2 : y_3], [x_0 : x_1 : x_2 : x_3]\big)$$

leaves X invariant and commutes with the $\mathfrak{S}_4$-action, which implies the result. □

Corollary 6.2.4. *If* $\lambda \neq 0$ *and* $\lambda^2 + 3\lambda + 18 \neq 0$*, then* $\mathrm{Aut}(X) \simeq \mathfrak{S}_4 \times \mathbb{Z}_2$.

Applying Corollary 6.1.4, we conclude that the Fano threefold X is K-stable.

6.2.2 $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$-invariant curve

Let $\widehat{G}$ be the subgroup in $\mathrm{GL}_4(\mathbb{C})$ generated by the matrices

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

and let G be the image of the group $\widehat{G}$ in $\mathrm{PGL}_4(\mathbb{C})$ via the natural projection $\mathrm{GL}_4(\mathbb{C}) \to \mathrm{PGL}_4(\mathbb{C})$. Then $\widehat{G} \simeq \mathbb{Z}_4.(\mathbb{Z}_4^2 \rtimes \mathbb{Z}_3)$ and $G \simeq \mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$, and their GAP ID's are [192,4] and [48,3], respectively. Using Linton (2007) and (Adem & Milgram, 2013, Corollary 5.4), one can check that $H^2(G, \mathbb{C}^*) \simeq \mathbb{Z}_4$ and $\widehat{G}$ is a covering group of the group G.

Lemma 6.2.5. *Let* G' *be a subgroup in* $\mathrm{PGL}_4(\mathbb{C})$ *such that* $G' \simeq G$ *and* G' *does not fix points in* $\mathbb{P}^3$*. Then* G' *is conjugate to* G *in* $\mathrm{PGL}_4(\mathbb{C})$.

Proof. The claim follows from (Cheltsov & Sarikyan, 2022, Lemma 2.7) and the classification of finite subgroups in $\mathrm{PGL}_4(\mathbb{C})$, which can be found in Blichfeldt (1917). Alternatively, one can prove the required assertion analyzing irreducible representations of the group $\widehat{G}$, which can be found in Dokchitser (n.d.). □

The main goal of this subsection is to show that the projective space $\mathbb{P}^3$ contains a G-invariant irreducible smooth non-hyperelliptic curve of degree 6 and genus 3, and this curve is unique up to the action of the normalizer of the group G in $\mathrm{PGL}_4(\mathbb{C})$. First, let us describe the normalizer. Set

$$C_4^{\pm} = \big\{ \big(1 \mp \sqrt{3}i\big)x_1^2 - (1 \pm \sqrt{3}i)x_2^2 + 2x_3^2 = 0, 2x_0^2 - (1 \pm \sqrt{3}i)x_1^2 - \big(1 \mp \sqrt{3}i\big)x_2^2 = 0 \big\} \subset \mathbb{P}^3.$$

Then $C_4^{\pm}$ is a G-invariant smooth elliptic curve, and

$$\mathrm{Aut}(\mathbb{P}^3, C_4^{\pm}) \simeq \mathrm{Aut}\Big(C_4^{\pm}, \big[\mathscr{O}_{\mathbb{P}^3}(1)\big|_{C_4^{\pm}}\big]\Big).$$

This implies that $\mathrm{Aut}(\mathbb{P}^3, C_4^{\pm})$ must be one of the following finite groups: $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_6$. On the other hand, one can check that $\mathrm{Aut}(\mathbb{P}^3, C_4^{\pm})$ contains the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This subgroup has GAP ID [96,72], and it is isomorphic to $\mathbb{Z}_2^3.\mathfrak{A}_4 \simeq \mathbb{Z}_4^2 \rtimes \mathbb{Z}_6$. Hence, we conclude that $\mathrm{Aut}(\mathbb{P}^3, C_4^{\pm}) \simeq \mathbb{Z}_2^3.\mathfrak{A}_4$. Let $G_{192,185}$ be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then its GAP ID is [192,185]. Note that $\mathrm{Aut}(\mathbb{P}^3, C_4^{\pm}) \lhd G_{192,185} \simeq \mathbb{Z}_2^3.\mathfrak{S}_4$ and $G \lhd G_{192,185}$.

Lemma 6.2.6. *The normalizer in $\mathrm{PGL}_4(\mathbb{C})$ of the subgroup G is the subgroup $G_{192,185}$.*

Proof. This follows from the fact that the curve $C_4^+ + C_4^-$ is $G_{192,185}$-invariant. □

Let us describe G-orbits in $\mathbb{P}^3$ of length less than 48. To do this, we let

$$\begin{aligned}
\Sigma_4 &= \mathrm{Orb}_G\big([1:0:0:0]\big), \\
\Sigma_{12} &= \mathrm{Orb}_G\big([1+i:\sqrt{2}:0:0]\big), \\
\Sigma'_{12} &= \mathrm{Orb}_G\big([1-i:\sqrt{2}:0:0]\big), \\
\Sigma_{16} &= \mathrm{Orb}_G\big([-1+\sqrt{3}i:-1-\sqrt{3}i:2:0]\big), \\
\Sigma'_{16} &= \mathrm{Orb}_G\big([-1-\sqrt{3}i:-1+\sqrt{3}i:2:0]\big), \\
\Sigma^u_{16} &= \mathrm{Orb}_G\big([1:1:1:u]\big) \text{ for } u \in \mathbb{C}, \\
\Sigma^t_{24} &= \mathrm{Orb}_G\big([2:t:0:0]\big) \text{ for } t \in \mathbb{C} \text{ such that } t \neq 0 \text{ and } t \neq \pm\sqrt{2} \pm \sqrt{2}i.
\end{aligned}$$

Then Σ_4, Σ_{12}, Σ'_{12}, Σ_{16}, Σ'_{16}, Σ^u_{16}, Σ^t_{24} are G-orbits of length 4, 12, 12, 16, 16, 16, 24, respectively.

Lemma 6.2.7. *Let Σ be a G-orbit in $\mathbb{P}^3$ such that $|\Sigma| < 48$. Then Σ is one of the G-orbits*

$$\Sigma_4,\ \Sigma_{12},\ \Sigma'_{12},\ \Sigma_{16},\ \Sigma'_{16},\ \Sigma^u_{16},\ \Sigma^t_{24},$$

where $u \in \mathbb{C}$ and $t \in \mathbb{C}$ such that $0 \neq t \neq \pm\sqrt{2} \pm \sqrt{2}i$.

Proof. Let us describe subgroups of the group G. To do this, identify the matrices M, N, A, B with their images in $\mathrm{PGL}_4(\mathbb{C})$. Set

$$C = ANBMA^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

Using the description of conjugacy classes of the subgroups of G, accessible with the GAP ID [48,3], for example on the webpage Dokchitser (n.d.), we see that all proper subgroups of the group G can be described as follows:

(i) $\langle B, C\rangle \simeq \mathbb{Z}_4^2$ is the unique (normal) subgroup of order 16,

(ii) $\langle A, M, N\rangle \simeq \mathfrak{A}_4$ is one of four conjugated subgroups of order 12,

(iii) $\langle B, M, N\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$ is one of three conjugated subgroups of order 8,

(iv) $\langle M, N\rangle \simeq \mathbb{Z}_2^2$ is the unique (normal) subgroup isomorphic to $\mathbb{Z}_2^2$,

(v) $\langle B\rangle \simeq \mathbb{Z}_4$ and $\langle CB\rangle \simeq \mathbb{Z}_4$ are non-conjugate subgroups. Their conjugacy classes consist of three subgroups, which are all subgroups of the group G isomorphic to $\mathbb{Z}_4$,

(vi) $\langle A\rangle \simeq \mathbb{Z}_3$ is one of sixteen conjugated subgroups of order 3,

(vii) $\langle M\rangle \simeq \mathbb{Z}_2$ is one of three conjugated subgroups of order 2.

Now, let Γ be the stabilizer in G of a point in Σ. Then Γ is a proper subgroup of the group G, since G fixes no points in $\mathbb{P}^3$. So, we may assume that Γ is one of the subgroups $\langle B, C\rangle$, $\langle A, M, N\rangle$, $\langle B, M, N\rangle$, $\langle M, N\rangle$, $\langle B\rangle$, $\langle CB\rangle$, $\langle A\rangle$, $\langle M\rangle$. On the other hand, one can check that

(i) $\langle B, C\rangle$ does not fix points in $\mathbb{P}^3$,

(ii) the only fixed point of $\langle A, M, N\rangle$ is the point $[1:0:0:0] \in \Sigma_4$,

(iii) $\langle B, M, N\rangle$ does not fix points in $\mathbb{P}^3$,

(iv) $\langle M, N\rangle$ does not fix points in $\mathbb{P}^3 \setminus \Sigma_4$,

(v) the only fixed point of $\langle B\rangle$ are the points

$$[1+i:\sqrt{2}:0:0], [1+i:-\sqrt{2}:0:0], [0:0:\sqrt{2}:1+i], [0:0:-\sqrt{2}:1+i],$$

which are contained in Σ_{12}, and the only fixed point of $\langle CB\rangle$ are the points

$$[\sqrt{2}:0:1-i:0], [-\sqrt{2}:0:1-i:0], [0:\sqrt{2}:0:1-i], [0:-\sqrt{2}:0:1-i],$$

which are contained in the G-orbit Σ'_{12},

(vi) the only fixed points of $\langle A\rangle$ are the points
- $[-1+\sqrt{3}i : -1-\sqrt{3}i : 2 : 0] \in \Sigma_{16}$,
- $[-1-\sqrt{3}i : -1+\sqrt{3}i : 2 : 0] \in \Sigma'_{16}$,
- $[1:1:1:t] \in \Sigma_{16}^t$ for any $t \in \mathbb{C}$,
- $[0:0:0:1] \in \Sigma_4$,

(vii) all fixed points of $\langle M\rangle$ are contained in the lines $\{x_0 = x_1 = 0\}$ and $\{x_2 = x_3 = 0\}$.

This implies the required assertion. □

Now, we are ready to present a G-invariant irreducible smooth curve in $\mathbb{P}^3$ of degree 6 and genus 3. For every $u \in \mathbb{C}$ such that $u \neq 0$, let $\mathscr{M}_3^u$ be the linear subsystem in $|\mathscr{O}_{\mathbb{P}^3}(3)|$ that consists of all cubic surfaces passing through the G-orbit Σ_{16}^u. If $u^4 = -3$, then the linear system $\mathscr{M}_3^u$ is 7-dimensional, and its base locus consists of one of the two elliptic curves C_4^+ or C_4^-. On the other hand, if $u^4 \neq -3$, then $\mathscr{M}_3^u$ is 3-dimensional. In this case, solving the corresponding system of 16 linear equations, we see that the base locus of the linear system $\mathscr{M}_3^u$ is given by the following equations:

$$f_1 = f_2 = f_3 = f_4 = 0, \tag{6}$$

where

$$\begin{aligned}
f_1 =&(u^4-1)x_3x_0^2+(u^4+3)x_0x_1x_2u+(u^4-1)x_3x_1^2-4x_3^3u^2+(u^4-1)x_2^2x_3=0,\\
f_2 =&(u^4-1)x_1x_0^2-u(u^4+3)x_0x_2x_3+4u^2x_1^3-(u^4-1)x_1x_2^2+(u^4-1)x_3^2x_1=0,\\
f_3 =&4u^2x_0^3-(u^4-1)x_0x_1^2+(u^4-1)x_0x_2^2+(u^4-1)x_3^2x_0-u(u^4+3)x_1x_2x_3=0,\\
f_4 =&(u^4-1)x_2x_0^2+u(u^4+3)x_0x_1x_3-(u^4-1)x_2x_1^2-4u^2x_2^3-(u^4-1)x_3^2x_2=0.
\end{aligned}$$

Using Maple or Mathematica, we can also eliminate variables x_2 and x_3 in (6). This gives

$$(u^{12}+17u^8+43u^4+3)F=0,$$

where F is non-zero polynomial in $\mathbb{C}[u,x_0,x_1]$ such that F is not divisible by any polynomial in $\mathbb{C}[u]$. Thus, if u is not a root of $u^{12}+17u^8+43u^4+3$, then the base locus of $\mathscr{M}_3^u$ is zero-dimensional, and one can check the converse is true. So, the base locus of $\mathscr{M}_3^u$ is zero-dimensional unless $u^4=-3$ or

$$u \in \left\{\frac{-1\pm\sqrt{3}}{2}+\frac{1\mp\sqrt{3}}{2}i, \frac{-1\pm\sqrt{3}}{2}+\frac{-1\pm\sqrt{3}}{2}i, \frac{1\pm\sqrt{3}}{2}+\frac{1\pm\sqrt{3}}{2}i, \frac{1\mp\sqrt{3}}{2}+\frac{-1\pm\sqrt{3}}{2}i\right\}.$$

Now, if $u=\frac{-1\pm\sqrt{3}}{2}+\frac{1\mp\sqrt{3}}{2}i$, $u=\frac{-1\pm\sqrt{3}}{2}+\frac{-1\pm\sqrt{3}}{2}i$, $u=\frac{1\pm\sqrt{3}}{2}+\frac{1\pm\sqrt{3}}{2}i$ or $u=\frac{1\mp\sqrt{3}}{2}+\frac{-1\pm\sqrt{3}}{2}i$, then the system of equations (6) defines an irreducible G-invariant smooth curve of degree 6 and genus 3. We will denote these curves by C_6, C_6', C_6'', C_6''', respectively. To be precise, we have

$$C_6 = \begin{cases}
(x_0^2+x_1^2+x_2^2)x_3-ix_3^3-(1-i)x_1x_0x_2=0,\\
(x_0^2-x_2^2+x_3^2)x_1+ix_1^3+(1-i)x_3x_0x_2=0,\\
(x_1^2-x_2^2-x_3^2)x_0-ix_0^3-(1-i)x_1x_3x_2=0,\\
(x_0^2-x_1^2-x_3^2)x_2-ix_2^3-(1-i)x_1x_3x_0=0,
\end{cases}$$

and C_6', C_6'', C_6''' can be obtained from C_6 applying elements of the normalizer $G_{192,185}$.

Fix $u = \frac{-1\pm\sqrt{3}}{2} + \frac{1\mp\sqrt{3}}{2}i$. Then (6) defines C_6. Choosing a different basis of the linear system $\mathscr{M}_3^u$, we obtain a birational map $\iota\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ given by $[x_0 : x_1 : x_2 : x_3] \mapsto [h_0 : h_1 : h_2 : h_3]$ for

$$\begin{aligned}
h_0 &= (1+i)x_3x_0x_2 - x_1^3 + ix_1(x_0^2 - x_2^2 + x_3^2),\\
h_1 &= (1+i)x_3x_1x_2 - x_0^3 - ix_0(x_1^2 - x_2^2 - x_3^2),\\
h_2 &= (1+i)x_3x_0x_1 - x_2^3 - ix_2(x_0^2 - x_1^2 - x_3^2),\\
h_3 &= (i-1)x_1x_0x_2 - ix_3^3 + x_3(x_0^2 + x_1^2 + x_2^2).
\end{aligned}$$

One can check that ι is a birational involution, and we have the following G-commutative diagram:

$$\begin{array}{ccc}
X & \xrightarrow{\ \tau\ } & X\\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi}\\
\mathbb{P}^3 & \underset{\iota}{\dashrightarrow} & \mathbb{P}^3
\end{array}$$

where π is the blow up of the curve C_6, and τ is an involution. Then X is a smooth Fano threefold in the deformation family №2.12, which can be defined as complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ given by

$$\begin{cases}
y_3x_0 - y_2x_0 + iy_2x_1 + y_3x_1 - y_0x_2 + iy_1x_2 + y_0x_3 + y_1x_3 = 0,\\
iy_0x_0 - y_1x_1 + y_3x_2 + y_2x_3 = 0,\\
y_2x_0 + y_3x_0 + iy_2x_1 - y_3x_1 - y_0x_2 - iy_1x_2 - y_0x_3 + y_1x_3 = 0,
\end{cases}$$

and π is induced by the projection to the first factor, where $([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3])$ are coordinates on $\mathbb{P}^3 \times \mathbb{P}^3$. Thus, in the notations in Section 6.1, we have

$$M_1 = \begin{pmatrix} 0 & 0 & -1 & 1\\ 0 & 0 & i & 1\\ -1 & i & 0 & 0\\ 1 & 1 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} i & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 1 & 1\\ 0 & 0 & i & -1\\ -1 & -i & 0 & 0\\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that M_1 and M_2 are symmetric, M_3 is skew-symmetric, and the involution τ is given by

$$\big([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]\big) \mapsto \big([y_0 : y_1 : y_2 : y_3], [x_0 : x_1 : x_2 : x_3]\big),$$

Corollary 6.2.8. *One has* $\mathrm{Aut}(\mathbb{P}^3, C_6) = G$ *and* $\mathrm{Aut}(X) \simeq \mathbb{Z}_4^2 \rtimes \mathfrak{S}_3$.

Proof. First, using the classification of automorphism groups of smooth curves of genus three, see Bars Cortina (2006); Dolgachev (2012), we see that C_6 is isomorphic to the Fermat quartic curve in $\mathbb{P}^2$. This can also be shown directly. Namely, it follows from Ottaviani (2024) that C_6 is isomorphic to the plane quartic curve

$$\left\{\det(xM_1+yM_2+zM_3)=0\right\}\subset\mathbb{P}^2_{x,y,z},$$

which is projectively isomorphic to the Fermat plane quartic curve.

We conclude that $\mathrm{Aut}(C_6)\simeq\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$. Therefore, if $\mathrm{Aut}(\mathbb{P}^3,C_6)\neq G$, then $\mathrm{Aut}(\mathbb{P}^3,C_6)\simeq\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$, and the subgroup $\mathrm{Aut}(\mathbb{P}^3,C_6)\subset\mathrm{PGL}_4(\mathbb{C})$ is contained in the normalizer of the group G in $\mathrm{PGL}_4(\mathbb{C})$, which is impossible since the normalizer is the group $G_{192,185}$ by Lemma 6.2.6, and $G_{192,185}$ does not contain subgroups isomorphic to $\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$.

Therefore, we conclude that $\mathrm{Aut}(\mathbb{P}^3,C_6)=G$. Now, one can explicitly check that $\langle G,\tau\rangle\simeq\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$, where we consider G as a subgroup in $\mathrm{Aut}(X)$. This gives $\mathrm{Aut}(X)\simeq\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$. □

By Corollary 6.1.4, the smooth Fano threefold X is K-stable.

In Section 6.4, we will see that X is the unique smooth Fano threefold in the family №2.12 whose automorphism group is isomorphic to the group $\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$. To do this, we need the following result:

Theorem 6.2.9. *The only G-invariant irreducible smooth curves in $\mathbb{P}^3$ of degree 6 are C_6, C_6', C_6'', C_6'''.*

Proof. Let C be a G-invariant irreducible smooth curve in $\mathbb{P}^3$ of degree 6, and let g be its genus. Then $g\leqslant 4$ by the Castelnuovo bound. Thus, it follows from Breuer (2000); Paulhus (2019) that either $g=1$, or $g=3$.

Note that $\Sigma_4\not\subset C$, because stabilizers in G of points in C are cyclic by (Flenner & Zaidenberg, 2005, Lemma 2.7).

Let $\Pi=\{x_3=0\}$, and let Γ be the the stabilizer of this plane in G. Then $\Gamma=\langle M,N,A\rangle\simeq\mathfrak{A}_4$, and all Γ-orbits in Π of length less than 12 can be described as follows:

1. $\Sigma_4\cap\Pi$ is the unique Γ-orbit of length 3,
2. $\Sigma_{12}\cap\Pi$ is a Γ-orbit of length 6,
3. $\Sigma_{12}'\cap\Pi$ is a Γ-orbit of length 6,
4. $\Sigma_{24}^t\cap\Pi$ consists in two Γ-orbits of length 6, where $0\neq t\neq\pm\sqrt{2}\pm\sqrt{2}i$,
5. $\Sigma_{16}\cap\Pi$, $\Sigma_{16}'\cap\Pi$ and $\Sigma_{16}^0\cap\Pi$ are Γ-orbits of length 4.

Thus, since $\Sigma_4\not\subset C$, $C\not\subset\Pi$, and $\Pi\cdot C$ is a Γ-invariant effective one-cycle of degree 6, we conclude that C contains at least one of the orbits Σ_{12} or Σ_{12}', and C does not contain Σ_{16}, Σ_{16}' and Σ_{16}^0.

If $g=1$, it follows from Breuer (2000); Paulhus (2019) that C does not contain G-orbits of length 12, which gives $g=3$. Then it follows from Breuer (2000); Paulhus (2019) that C contains two G-orbits of length 16, so $\Sigma_{16}^{t} \subset C$ for some $t \neq 0$.

Using the classification of automorphism groups of smooth curves of genus three, see Bars Cortina (2006); Dolgachev (2012), we see that the curve C is isomorphic to the Fermat quartic curve in $\mathbb{P}^2$. Hence, the curve C is not hyperelliptic.

Let $\mathscr{M}_3$ be the linear subsystem in $|\mathscr{O}_{\mathbb{P}^3}(3)|$ that consists of all cubic surfaces passing through C. Then $\mathscr{M}_3$ is three-dimensional, and the curve C is its base locus by Homma (1980), because C is not hyperelliptic. Therefore, using the notations introduced earlier, we see that $\mathscr{M}_3 = \mathscr{M}_3^t$ for an appropriate $t \in \mathbb{C}$. Now, arguing as above, we see that

$$t \in \left\{ \frac{-1\pm\sqrt{3}}{2} + \frac{-1\pm\sqrt{3}}{2}i, \frac{-1\pm\sqrt{3}}{2} + \frac{1\mp\sqrt{3}}{2}i, \frac{1\pm\sqrt{3}}{2} + \frac{1\pm\sqrt{3}}{2}i, \frac{1\mp\sqrt{3}}{2} + \frac{-1\pm\sqrt{3}}{2}i \right\},$$

which implies that C is one of the curves C_6, C_6', C_6'', C_6''' as claimed. □

6.2.3 Curves over $\mathbb{Q}$ without rational points

Let us use notations introduced in Section 6.1. Suppose, in addition, that

$$C = \{x^4 + xyz^2 + y^4 + y^3z - 31yz^3 + 4z^4 = 0\} \subset \mathbb{P}^2_{x,y,z}.$$

Then C is smooth. One can show that $C(\mathbb{Q}) = \varnothing$ using the reduction modulo 3. Set

$$P_1 = [1-i:0:1], P_2 = [1+i:0:1], P_3 = [-1+i:0:1], P_4 = [-1-i:0:1].$$

and $D = 3(P_3+P_4) - K_C$. Then D is defined over $\mathbb{Q}$ and D satisfies (1). The later condition can be checked using the following Magma code:

```
Q<i>:=QuadraticField(-1);
P2<x,y,z>:=ProjectiveSpace(Q,2);
X:=Scheme(P2,[x^4+x*y*z^2+y^4+y^3*z-31*y*z^3+4*z^4]);
C:=Curve(X);
P3:=C![-1-i,0,1];
P4:=C![-1+i,0,1];
D3:=Divisor(P3);
D4:=Divisor(P4);
D:=3*D3+3*D4-CanonicalDivisor(C);
Dimension(D);
```

Then C_6 is defined over $\mathbb{Q}$, and it is isomorphic to C over $\mathbb{Q}$. In particular, the curve C_6 does not contain $\mathbb{Q}$-rational points. Hence, by Corollary 6.1.7, the smooth Fano threefold X is K-stable.

One can explicitly find defining equations of C_6 as follows. Let $\mathscr{M}$ be the linear system of cubic curves in $\mathbb{P}^2$ whose general member is tangent to C with multiplicity 3 at the points P_1 and P_2. Then

$$\mathscr{M}\big|_C = 3P_1 + 3P_2 + |3(P_3 + P_4)|.$$

Thus, to compute the embedding $C \hookrightarrow \mathbb{P}^3$, it is enough to find a basis of the linear system $\mathscr{M}$, which can be done using linear algebra. After this, it is easy to find defining equations of the curve C_6.

6.2.4 Real pointless threefolds

Now, we explain how to construct real smooth Fano threefolds in the deformation family №2.12 that do not have real points. By Corollary 6.1.8, all of them are K-stable. We start with the following example.

Example 45. Let U be a three-dimensional Severi–Brauer variety defined over $\mathbb{R}$ such that $U \not\simeq \mathbb{P}^3_{\mathbb{R}}$. Recall from Gille and Szamuely (2017); Kollár (2016) that U exists, it is unique, and, in particular, it is isomorphic to its dual variety. Set $W = U \times U$. Then

$$W_{\mathbb{C}} \simeq \mathbb{P}^3 \times \mathbb{P}^3.$$

Since $U \simeq U^\vee$, the Picard group $\mathrm{Pic}_{\mathbb{R}}(W)$ contains a real line bundle L such that $L_{\mathbb{C}}$ has degree $(1,1)$. Let V be any smooth complete intersection of three divisors in $|L|$. Then V is a smooth Fano threefold in the family №2.12, and V does not have real points, because W does not have real points.

Let us present another, more explicit, construction of pointless real smooth Fano threefolds in the deformation family №2.12. For a point $P = ([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]) \in \mathbb{P}^3 \times \mathbb{P}^3$, let us consider the symmetric matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}$$

and the skew-symmetric matrix

$$B = \begin{pmatrix} 0 & b_{01} & b_{02} & b_{03} \\ -b_{01} & 0 & b_{12} & b_{13} \\ -b_{02} & -b_{12} & 0 & b_{23} \\ -b_{03} & -b_{13} & -b_{23} & 0 \end{pmatrix}$$

defined (up to a common scalar multiple) as follows:

$$a_{nm} = \frac{x_n y_m + x_m y_n}{2}$$

and

$$b_{nm} = \frac{x_n y_m - x_m y_n}{2i}$$

for every $n \in \{0,1,2,3\}$ and $m \in \{0,1,2,3\}$ such that $n \neq m$, and $a_{nn} = x_n y_n$ for each $n \in \{0,1,2,3\}$. Set $M = A + iB$. Then

$$M = \begin{pmatrix} x_0y_0 & x_0y_1 & x_0y_2 & x_0y_3 \\ x_1y_0 & x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_0 & x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_0 & x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix}.$$

Therefore, we see that the constructed map $P \mapsto M$ gives us the Serge embedding $\mathbb{P}^3 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^{15}$, where we consider $\mathbb{P}^{15}$ as a projectivization of the vector space of all 4×4 matrices.

Now, we consider matrices A and B on their own, and we also assume that all a_{nm} and b_{nm} are real. Then M is a Hermitian 4×4 matrix. Projectivizing the vector space of Hermitian 4×4 matrices, we obtain $\mathbb{P}^{15}_{\mathbb{R}}$ with coordinates $[a_{00} : a_{01} : \cdots : b_{13} : b_{23}]$. Let us consider M as a point in $\mathbb{P}^{15}_{\mathbb{R}}$, and set

$$V = \left\{ M \in \mathbb{P}^{15}_{\mathbb{R}} \;\middle|\; \operatorname{rank}(M) \leqslant 1 \right\} \subset \mathbb{P}^{15}_{\mathbb{R}}.$$

Then V is a real projective subvariety in $\mathbb{P}^{15}_{\mathbb{R}}$. Moreover, over $\mathbb{C}$, the subvariety $V_{\mathbb{C}}$ is the image of the map $P \mapsto M$ constructed above, which implies that $V_{\mathbb{C}} \simeq \mathbb{P}^3 \times \mathbb{P}^3$, so V is a form of $\mathbb{P}^3_{\mathbb{R}} \times \mathbb{P}^3_{\mathbb{R}}$. But $V \not\simeq \mathbb{P}^3_{\mathbb{R}} \times \mathbb{P}^3_{\mathbb{R}}$ over $\mathbb{R}$, because V is the Weil restriction of $\mathbb{P}^3$ over the reals (cf.(Gorchinskiy & Shramov, 2015, Exercise 8.1.6)), which implies that $V(\mathbb{R}) \neq \varnothing$, and $\mathrm{Pic}_{\mathbb{R}}(V)$ is generated by the class of a hyperplane section.

Now, let H_1, H_2, H_3 be three real hyperplane sections of $V \subset \mathbb{P}^{15}_{\mathbb{R}}$. Set $X = H_1 \cap H_2 \cap H_3$. Suppose that X is smooth and three-dimensional. Then X is a real form of a smooth Fano threefold in the deformation family №2.12 such that $\mathrm{Pic}_{\mathbb{R}}(X) = \mathbb{Z}[-K_X]$. Moreover, Corollary 6.1.8 gives the following.

Corollary 6.2.10. *If X does not have real points, then X is K-stable.*

Such smooth Fano threefolds without real points do exist:

Example 46. Suppose that H_1 is cut out by $a_{00} + a_{11} + a_{22} + a_{33} = 0$. Then H_1 is smooth, because its preimage in $\mathbb{P}^3 \times \mathbb{P}^3$ via the map constructed above is given by

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0.$$

We claim the fivefold H_1 does not have real points. It will be sufficient to show that if $M \in V$, then the corresponding real numbers a_{00}, a_{11}, a_{22}, a_{33} are either all non-negative or all non-positive and cannot be all zero. To this end, note that for any $n < m$, by the rank assumption on M we have

$$a_{nn}a_{mm} = (a_{nm} + ib_{nm})(a_{nm} - ib_{nm}) = |a_{nm}|^2 + |b_{nm}|^2 \geq 0.$$

In particular, a_{nn} and a_{mm} cannot have differing signs. Moreover, if all the a_{nn} were zero, then that would imply $|a_{nm}|^2 + |b_{nm}|^2 = 0$ for all $n < m$ too, which in turn implies $M = 0$. This cannot happen. Thus, they cannot be all zero.

Similarly, set $H_2 = \{a_{03} + 2a_{12} = 0\} \cap V$ and $H_3 = \{a_{02} + a_{13} + a_{23} = 0\} \cap V$. Then $V_{\mathbb{C}}$ is isomorphic to the complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ given by

$$\begin{cases} x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0, \\ x_0y_3 + x_3y_0 + 2x_1y_2 + 2x_2y_1 = 0, \\ x_0y_2 + x_2y_0 + x_3y_1 + x_1y_3 + x_2y_3 + x_3y_2 = 0. \end{cases}$$

This complete intersection is a smooth threefold, so X is smooth, and it has no real points, because the divisor H_1 does not have real points.

6.3 The proof of Theorem A

Let us use all the notation and assumptions introduced in Section 6.1. To start with, we will present some results from Abban and Zhuang (2022); Araujo et al. (2023a) that will be used in the proof of Theorem A. Let $\mathbf{F}$ be a prime divisor over X, and let Z be its center on X. Suppose that

- either Z is a point,
- or Z is an irreducible curve.

Let P be any point in Z. Choose an irreducible smooth surface $S \subset X$ such that $P \in S$. Set

$$\tau = \sup\Big\{ u \in \mathbb{Q}_{\geqslant 0} \;\Big|\; \text{the divisor } -K_X - uS \text{ is pseudo-effective} \Big\}.$$

For $u \in [0, \tau]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_X - uS$, and let $N(u)$ be its negative part. Then $\beta(S) = 1 - S_X(S)$, where

$$S_X(S) = \frac{1}{-K_X^3} \int\limits_0^\infty \operatorname{vol}\big(- K_X - uS\big) du = \frac{1}{20} \int\limits_0^\tau P(u)^3 du.$$

Let us show how to compute $P(u)$ and $N(u)$. Set $H=\pi^*(\mathscr{O}_{\mathbb{P}^3}(1))$ and $H'=(\pi')^*(\mathscr{O}_{\mathbb{P}^3}(1))$. Then

$$H\sim 3H'-E', E\sim 8H'-3E', H'\sim 3H-E, E'\sim 8H-3E,$$

where E and E' are exceptional surfaces of the blow ups π and π', respectively. Since $\rho(X)=2$, the Mori cone is spanned by two extremal rays, and it is clear that the rays spanned by the fibers of $E\to C_6$ and $E'\to C_6'$, say ℓ and ℓ' do the trick. We have

$$H.\ell=0, E.\ell=-1, H'.\ell=1, E'.\ell=3.$$

With this in mind:

Example 47. Suppose that $S\in|H|$. Then $\tau=\frac{4}{3}$. Moreover, by considering intersection with ℓ' we have

$$P(u)\sim_{\mathbb{R}}\begin{cases}(4-u)H-E \text{ for } 0\leqslant u\leqslant 1,\\ (4-3u)H' \text{ for } 1\leqslant u\leqslant \dfrac{4}{3},\end{cases}$$

and

$$N(u)=\begin{cases}0 \text{ for } 0\leqslant u\leqslant 1,\\ (u-1)E' \text{ for } 1\leqslant u\leqslant \dfrac{4}{3},\end{cases}$$

which gives $S_X(S)=\frac{1}{20}\int\limits_0^{\frac{4}{3}}\big(P(u)\big)^3du=\frac{1}{20}\int\limits_0^{1}(2-u)(u^2-10u+10)du+\frac{1}{20}\int\limits_1^{\frac{4}{3}}(4-3u)^3du=\frac{53}{120}$.

Example 48. Suppose that $S=E$. Then $\tau=\frac{1}{2}$,

$$P(u)\sim_{\mathbb{R}}\begin{cases}4H-(1+u)E \text{ for } 0\leqslant u\leqslant \dfrac{1}{3},\\ (4-8u)H' \text{ for } \dfrac{1}{3}\leqslant u\leqslant \dfrac{1}{2},\end{cases}$$

and

$$N(u)=\begin{cases}0 \text{ for } 0\leqslant u\leqslant \dfrac{1}{3},\\ (3u-1)E' \text{ for } \dfrac{1}{3}\leqslant u\leqslant \dfrac{1}{2},\end{cases}$$

which gives $S_X(S)=\frac{1}{20}\int\limits_0^{\frac{1}{2}}4(1-u)(5-7u^2-10u)du+\frac{1}{20}\int\limits_{\frac{1}{2}}^{\frac{1}{3}}64(1-2u)^3du=\frac{11}{60}$.

Now, we choose an irreducible curve $C\subset S$ that contains the point P. For instance, if Z is a curve, and S contains Z, then we can choose $C=Z$. Since $S\not\subset\mathrm{Supp}(N(u))$, we can write

$$N(u)\big|_S=d(u)C+N'(u),$$

where $d(u) = \mathrm{ord}_C(N(u)|_S)$, and $N'(u)$ is an effective $\mathbb{R}$-divisor on S such that $C \not\subset \mathrm{Supp}(N'(u))$. Now, for every $u \in [0, \tau]$, we set

$$t(u) = \sup\Big\{ v \in \mathbb{R}_{\geqslant 0} \ \Big|\ \text{the divisor } P(u)\big|_S - vC \text{ is pseudo-effective} \Big\}.$$

For $v \in [0, t(u)]$, we let $P(u, v)$ be the positive part of the Zariski decomposition of $P(u)|_S - vC$, and we let $N(u, v)$ be its negative part. Following Abban and Zhuang (2022); Araujo et al. (2023a), we let

$$S\big(W^S_{\bullet,\bullet}; C\big) = \frac{3}{(-K_X)^3} \int\limits_0^\tau d(u) \Big(P(u)\big|_S\Big)^2 du + \frac{3}{(-K_X)^3} \int\limits_0^\tau \int\limits_0^\infty \mathrm{vol}\big(P(u)\big|_S - vC\big) dv du,$$

which we can rewrite as

$$S\big(W^S_{\bullet,\bullet}; C\big) = \frac{3}{(-K_X)^3} \int\limits_0^\tau d(u) \Big(P(u)\big|_S\Big)^2 du + \frac{3}{(-K_X)^3} \int\limits_0^\tau \int\limits_0^{t(u)} \big(P(u,v)\big)^2 dv du.$$

If Z is a curve, $Z \subset S$ and $C = Z$, then it follows from (Araujo et al., 2023b, Corollary 1.110), which is based on Abban and Zhuang (2022), that

$$\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{ \frac{1}{S_X(S)}, \frac{1}{S\big(W^S_{\bullet,\bullet}; C\big)} \right\}. \tag{7}$$

Let $f\colon \widetilde{S} \to S$ be the blow up of the point P, let F be the f-exceptional curve, let $\widetilde{N}'(u)$ be the strict transform on $\widetilde{S}$ of the $\mathbb{R}$-divisor $N(u)|_S$, and let $\widetilde{d}(u) = \mathrm{mult}_P(N(u)|_S)$. Then

$$f^*\big(N(u)\big|_S\big) = \widetilde{d}(u) F + \widetilde{N}'(u).$$

For every $u \in [0, \tau]$, set

$$\widetilde{t}(u) = \sup\Big\{ v \in \mathbb{R}_{\geqslant 0} \ \Big|\ \text{the divisor } f^*\big(P(u)\big|_S\big) - vF \text{ is pseudo-effective} \Big\}.$$

For $v \in [0, \widetilde{t}(u)]$, we let $\widetilde{P}(u, v)$ be the positive part of the Zariski decomposition of $f^*(P(u)|_S) - vF$, and let $\widetilde{N}(u, v)$ be its negative part. Let

$$S\big(W^S_{\bullet,\bullet}; F\big) = \frac{3}{(-K_X)^3} \int\limits_0^\tau \widetilde{d}(u) \Big(f^*\big(P(u)\big|_S\big)\Big)^2 du + \frac{3}{(-K_X)^3} \int\limits_0^\tau \int\limits_0^\infty \mathrm{vol}\big(f^*\big(P(u)\big|_S\big) - vF\big) dv du.$$

Then

$$S\big(W^S_{\bullet,\bullet}; F\big) = \frac{3}{(-K_X)^3} \int\limits_0^\tau \widetilde{d}(u) \big(\widetilde{P}(u,0)\big)^2 du + \frac{3}{20} \int\limits_0^\tau \int\limits_0^{\widetilde{t}(u)} \big(\widetilde{P}(u,v)\big)^2 dv du.$$

For every point $O \in F$, we let

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big) = \frac{3}{(-K_X)^3}\int\limits_0^{\tau}\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\cdot F\big)^2 dvdu + F_O\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}\big)$$

for

$$F_O\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}\big) = \frac{6}{(-K_X)^3}\int\limits_0^{\tau}\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\cdot F\big)\cdot \mathrm{ord}_O\big(\widetilde{N}'(u)\big|_F + \widetilde{N}(u,v)\big|_F\big)dvdu.$$

Then it follows from (Araujo et al., 2023b, Remark 1.113) that

$$\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(S)}, \frac{2}{S\big(W^S_{\bullet,\bullet};F\big)}, \inf_{O\in F}\frac{1}{S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)}\right\}. \tag{8}$$

Thus, if $S_X(S) < 1$, $S(W^S_{\bullet,\bullet};F) < 2$ and $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O) < 1$ for every point $O \in F$, then $\beta(\mathbf{F}) > 0$.

Now, we are ready to prove Theorem A. We must show that $\beta(\mathbf{F}) > 0$ if Z is not a point in $E \cap E'$. If Z is a surface, it follows from (Fujita, 2016, Theorem 10.1) that $\beta(\mathbf{F}) > 0$. Hence, we may assume that Z is **not** a surface.

Lemma 6.3.1 (cf. Cheltsov and Pokora (2023))**.** *Suppose that Z is a curve, $Z \subset E$, and $\pi(Z)$ is not a point. Then $\beta(\mathbf{F}) > 0$.*

Proof. Let e be the invariant of the ruled surface E defined in Proposition 2.8 in (Hartshorne, 2013, Chapter V). Then $e \geqslant -3$, by Nagata (1970). Moreover, there exists a section C_0 of the projection $E \to C_6$ such that $C_0^2 = -e$. Let ℓ a fiber of this projection. Then $H|_E \equiv 6\ell$ and $E|_E \equiv -C_0 + \lambda\ell$ for some integer λ. Since

$$-28 = -c_1\big(N_{C_6/\mathbb{P}^3}\big) = E^3 = (-C_0+\lambda\ell)^2 = -e-2\lambda,$$

we get $\lambda = \frac{28-e}{2}$, so e is even and $e \geqslant -2$. Since H' is nef and $H'|_E \equiv C_0 + (18-\lambda)\ell$, we get

$$0 \leqslant H'\cdot C_0 = \big(C_0 + (18-\lambda)\ell\big)\cdot C_0 = \frac{8-e}{2},$$

which implies that $e \leqslant 8$. Thus, we see that $e \in \{-2,0,2,4,6,8\}$.

Set $S = E$ and $C = Z$. Let us estimate $S(W^S_{\bullet,\bullet};C)$. It follows from Example 48 that $\tau = \frac{1}{2}$ and

$$P(u)\big|_S \equiv \begin{cases} (1+u)C_0 + \dfrac{20+e+ue-28u}{2}\ell \ \text{ for } 0 \leqslant u \leqslant \dfrac{1}{3}, \\ (4-8u)C_0 + 2(1-2u)(8+e)\ell \ \text{ for } \dfrac{1}{3} \leqslant u \leqslant \dfrac{1}{2}. \end{cases}$$

If $0 \leqslant u \leqslant \frac{1}{2}$, then $N(u)=0$. If $\frac{1}{2} \leqslant u \leqslant \frac{1}{3}$, then $N(u)\big|_S=(3u-1)E'|_S$, where $E'|_S \equiv 3C_0+\frac{12+3e}{2}\ell$. By Proposition 2.20 in (Hartshorne, 2013, Chapter V), we have $Z \equiv aC_0+b\ell$ for integers a and b such that $a \geqslant 0$ and $b \geqslant ae$. Since $\pi(Z)$ is not a point, we have $a \geqslant 1$. Then $\mathrm{ord}_C(E'|_S) \leqslant 3$, as can be seen by intersecting with ℓ. Hence, if $\frac{1}{3} \leqslant u \leqslant \frac{1}{2}$, then $d(u) \leqslant 3(3u-1)$. This gives

$$S(W^S_{\bullet,\bullet};C)=\frac{3}{20}\int\limits_{\frac{1}{3}}^{\frac{1}{2}}128(2u-1)^2d(u)du+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu \leqslant$$

$$\leqslant \frac{3}{20}\int\limits_{\frac{1}{3}}^{\frac{1}{2}}384(3u-1)(2u-1)^2du+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu=$$

$$=\frac{2}{45}+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu=\frac{2}{45}+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-v(aC_0+b\ell)\big)dvdu.$$

Thus, we conclude that $S(W^S_{\bullet,\bullet};C) \leqslant \frac{2}{45}+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-v(aC_0+b\ell)\big)dvdu.$

Suppose that $b \geqslant 0$. Then

$$\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-v(aC_0+b\ell)\big)dvdu \leqslant \frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)|_S-vC_0\big)dvdu.$$

On the other hand, we have

$$P(u)\big|_S-vC_0 \equiv \begin{cases}(1+u-v)C_0+\dfrac{20+e+ue-28u}{2}\ell \text{ if } 0 \leqslant u \leqslant \dfrac{1}{3},\\ (4-8u-v)C_0+2(1-2u)(8+e)\ell \text{ if } \dfrac{1}{3} \leqslant u \leqslant \dfrac{1}{2}.\end{cases}$$

Hence, if $0 \leqslant u \leqslant \frac{1}{3}$, then the divisor $P(u)\big|_S - vC_0$ is pseudoeffective $\iff$ it is nef $\iff$ $v \leqslant 1+u$. Likewise, if $\frac{1}{3} \leqslant u \leqslant \frac{1}{2}$, then $P(u)\big|_S - vC_0$ is pseudoeffective $\iff$ it is nef $\iff$ $v \leqslant 4-8u$. Then

$$S(W^S_{\bullet,\bullet};C) \leqslant \frac{2}{45} + \frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty} \operatorname{vol}\big(P(u)|_S - vC_0\big)dvdu =$$

$$= \frac{2}{45} + \frac{3}{20}\int\limits_0^{\frac{1}{3}}\int\limits_0^{1+u}\left((1+u-v)C_0 + \frac{20+e+ue-28u}{2}\ell\right)^2 dvdu +$$

$$+\frac{3}{20}\int\limits_{\frac{1}{3}}^{\frac{1}{2}}\int\limits_0^{4-8u}\big((4-8u-v)C_0 + 2(1-2u)(8+e)\ell\big)^2 dvdu = \frac{23e}{1440} + \frac{221}{360} < 1,$$

because $e \leqslant 8$. Then $\beta(\mathbf{F}) > 0$ by (7), since we know from Example 48 that $S_X(S) < 1$.

Thus, to complete the proof, we may assume that $b < 0$. Then $e < 0$, so that $e = -2$, since $b \geqslant ae$. Hence, from Proposition 2.21 in (Hartshorne, 2013, Chapter V) it follows that $a \geqslant 2$ and $b \geqslant -a$. Then

$$\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\operatorname{vol}\big(P(u)|_S - vC\big)dvdu \leqslant \frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\operatorname{vol}\big(P(u)|_S - v(2C_0 - 2\ell)\big)dvdu.$$

Moreover, arguing as above, we compute

$$\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\operatorname{vol}\big(P(u)|_S - v(2C_0-2\ell)\big)dvdu = \frac{41}{144},$$

which gives $S(W^S_{\bullet,\bullet};C) \leqslant \frac{2}{45} + \frac{41}{144} = \frac{79}{240} < 1$, so that $\beta(\mathbf{F}) > 0$ by (7). □

Similarly, we prove that

Lemma 6.3.2. *Suppose that Z is a curve, $Z \subset E'$, and $\pi'(Z)$ is not a point. Then $\beta(\mathbf{F}) > 0$.*

Now, suppose that Z is **not** a point in $E \cap E'$. To prove Theorem A, we must show that $\beta(\mathbf{F}) > 0$. Let P be a general point in Z. By Lemmas 6.3.1 and 6.3.2, we may assume that either $P \notin E$ or $P \notin E'$. Hence, without loss of generality, we may assume that $P \notin E$. Let us show that $\beta(\mathbf{F}) > 0$.

Let S be a sufficiently general surface in $|H|$ that contains P. Then it follows from the adjunction formula that $-K_S \sim H'|_S$. Set $\Pi = \pi(S)$. Then Π is a general plane in $\mathbb{P}^3$ that contains $\pi(P)$. Write

$$\Pi \cap C_6 = \big\{P_1, P_2, P_3, P_4, P_5, P_6\big\},$$

where $P_1, P_2, P_3, P_4, P_5, P_6$ are distinct points. Then π induces a birational morphism $\varpi\colon S \to \Pi$, which is a blow up of the intersection points P_1, P_2, P_3, P_4, P_5, P_6.

Lemma 6.3.3. *The divisor $-K_S$ is ample.*

Proof. We must show that at most two points among P_1, P_2, P_3, P_4, P_5, P_6 are contained in a line, and not all of these six points are contained in an irreducible conic.

If there exists a line $\ell \subset \Pi$ such that ℓ contains at least three points among P_1, P_2, P_3, P_4, P_5, P_6, then ℓ is a trisecant of the curve C_6, so that the line ℓ is contained in $\pi(E')$, and its strict transform on the threefold X is a fiber of the projection $E' \to C_6'$. But the planes in $\mathbb{P}^3$ containing $\pi(P)$ and a trisecant of the curve C_6 form a one-dimensional family. Hence, a general plane in $\mathbb{P}^3$ that contains the point $\pi(P)$ does not contain trisecants of the curve C_6. Therefore, we conclude that at most two points among P_1, P_2, P_3, P_4, P_5, P_6 are contained in a line.

Similarly, if the points P_1, P_2, P_3, P_4, P_5, P_6 are contained in an irreducible conic in Π, then its strict transform on the threefold X has trivial intersection with $H' \sim 3H - E$, which implies that this conic is the image of a fiber of the projection $E' \to C_6'$, which is impossible, since these fibers are mapped to lines in $\mathbb{P}^3$. Therefore, the divisor $-K_S$ is ample. □

Thus, we can identify S with a smooth cubic surface in $\mathbb{P}^3$. Recall that $P \notin E$.

Lemma 6.3.4. *Suppose that there exists a line $\ell \subset S$ such that $P \in \ell$. Then $\pi(\ell)$ is a conic.*

Proof. If $\pi(\ell)$ is not a conic, then $\pi(\ell)$ is a secant of the curve C_6 that contains $\pi(P)$. Let us show that we can choose Π such that it does not contain any secant of the curve C_6.

Let $\varphi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the linear projection from $\pi(P)$. Since C_6 is not hyperelliptic and $\pi(P) \notin C_6$, one of the following two possibilities holds:

1. $\varphi(C_6)$ is a singular curve of degree 6, and φ induces a birational morphism $C_6 \to \varphi(C_6)$,
2. $\varphi(C_6)$ is a smooth cubic, and φ induces a double cover $C_6 \to \varphi(C_6)$.

In the second case, the curve C_6 is contained in an irrational cubic cone in $\mathbb{P}^3$, which is impossible, because the composition $\pi' \circ \pi^{-1}$ birationally maps every cubic surface containing C_6 to a plane in $\mathbb{P}^3$. Thus, we see that $\varphi(C_6)$ is a singular irreducible curve of degree 6.

All secants of the curve C_6 containing $\pi(P)$ are mapped by φ to singular points of the curve $\varphi(C_6)$. Since this curve has finitely many singular points, there are finitely many secants of the curve C_6 that pass through $\pi(P)$. Hence, since Π is a general plane in $\mathbb{P}^3$ that contains $\pi(P)$, we may assume that it does not contain secants of the curve C_6 containing $\pi(P)$, so $\pi(\ell)$ is a conic. □

Let T be the unique hyperplane section of the surface $S \subset \mathbb{P}^3$ that is singular at P. Then it follows from Lemma 6.3.4 that either P is not contained in any line in S, and one of the following cases holds:

(a) T is an irreducible cubic curve that has a node at P;

(b) T is an irreducible cubic curve that has a cusp at P;

or P is contained in a unique line $\ell \subset S$, $\pi(\ell)$ is a conic, and one of the following cases holds:

(c) $T = \ell + C_2$ for a smooth conic C_2 that intersect ℓ transversally at P;

(d) $T = \ell + C_2$ for a smooth conic C_2 that is tangent to ℓ at P.

Let us construct another curve in S that is also singular at P. Namely, for each $i \in \{1,2,3,4,5,6\}$, let ℓ_i be the proper transform on S of the unique line in Π that passes through the points $\pi(P)$ and P_i. Set $L = \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 + \ell_6$. Then it follows from Example 47 that

$$P(u)\big|_S \sim_{\mathbb{R}} \begin{cases} \dfrac{2+u}{3}T + \dfrac{1-u}{3}L \text{ if } 0 \leqslant u \leqslant 1, \\ (4-3u)T \text{ if } 1 \leqslant u \leqslant \dfrac{4}{3}. \end{cases}$$

Recall from Example 47 that $\tau = \frac{4}{3}$ and $S_X(S) = \frac{53}{120}$.

Let $\widetilde{T}$ and $\widetilde{L}$ be the proper transforms on $\widetilde{S}$ of the curves T and L, respectively. If $0 \leqslant u \leqslant 1$, then

$$f^*\big(P(u)|_S\big) - vF \sim_{\mathbb{R}} \frac{2+u}{3}\widetilde{T} + \frac{1-u}{3}\widetilde{L} + \frac{10-4u-3v}{3}F,$$

which implies that $\widetilde{t}(u) = \frac{10-4u}{3}$. Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$, then

$$f^*\big(P(u)|_S\big) - vF \sim_{\mathbb{R}} (4-3u)\widetilde{T} + (8-6u-v)F,$$

which implies that $\widetilde{t}(u) = 8-6u$.

Finally, set $R = E'|_S$. Then R is a smooth curve. Let $\widetilde{R}$ be its strict transform on $\widetilde{S}$. Then

$$\widetilde{N}'(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)\widetilde{R} \text{ if } 1 \leqslant u \leqslant \dfrac{4}{3}. \end{cases}$$

So, if $0 \leqslant u \leqslant 1$ or $P \notin E'$, then $\widetilde{d}(u) = 0$. Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$ and $P \in E'$, then $\widetilde{d}(u) = u-1$.

Lemma 6.3.5. *Suppose that P is not contained in any line in S. Then $\beta(\mathbf{F}) > 0$.*

Proof. The curve T is irreducible. If $0 \leqslant u \leqslant 1$, then by considering intersection with $\widetilde{T}$ and $\widetilde{L}$, we obtain:

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} \dfrac{2+u}{3}\widetilde{T} + \dfrac{1-u}{3}\widetilde{L} + \dfrac{10-4u-3v}{3}F \text{ if } 0 \leqslant v \leqslant \dfrac{6-3u}{2}, \\ \dfrac{20-8u-6v}{3}\widetilde{T} + \dfrac{1-u}{3}\widetilde{L} + \dfrac{10-4u-3v}{3}F \text{ if } \dfrac{6-3u}{2} \leqslant v \leqslant 3-u, \\ \dfrac{10-4u-3v}{3}\big(2\widetilde{T} + \widetilde{L} + F\big) \text{ if } 3-u \leqslant v \leqslant \dfrac{10-4u}{3}, \end{cases}$$

and

$$N(u,v)=\begin{cases}0 \text{ if } 0\leqslant v\leqslant \dfrac{6-3u}{2},\\ (2v-6+3u)\widetilde{T} \text{ if } \dfrac{6-3u}{2}\leqslant v\leqslant 3-u,\\ (2v-6+3u)\widetilde{T}+(v+u-3)\widetilde{L} \text{ if } 3-u\leqslant v\leqslant \dfrac{10-4u}{3}.\end{cases}$$

This gives

$$\big(P(u,v)\big)^2=\begin{cases}u^2-v^2-8u+10 \text{ if } 0\leqslant v\leqslant \dfrac{6-3u}{2},\\ 10u^2+12uv+3v^2-44u-24v+46 \text{ if } \dfrac{6-3u}{2}\leqslant v\leqslant 3-u,\\ (10-4u-3v)^2 \text{ if } 3-u\leqslant v\leqslant \dfrac{10-4u}{3}\end{cases}$$

and

$$P(u,v)\cdot F=\begin{cases}v \text{ if } 0\leqslant v\leqslant \dfrac{6-3u}{2},\\ 12-6u-3v \text{ if } \dfrac{6-3u}{2}\leqslant v\leqslant 3-u,\\ 30-12u-9v \text{ if } 3-u\leqslant v\leqslant \dfrac{10-4u}{3}.\end{cases}$$

Similarly, if $1\leqslant u\leqslant \frac{4}{3}$, then

$$P(u,v)\sim_{\mathbb{R}}\begin{cases}(4-3u)\widetilde{T}+(8-6u-v)F \text{ if } 0\leqslant v\leqslant \dfrac{12-9u}{2},\\ (8-6u-v)\big(2\widetilde{T}+F\big) \text{ if } \dfrac{12-9u}{2}\leqslant v\leqslant 8-6u,\end{cases}$$

and

$$N(u,v)=\begin{cases}0 \text{ if } 0\leqslant v\leqslant \dfrac{12-9u}{2},\\ (2v+9u-12)\widetilde{T} \text{ if } \dfrac{12-9u}{2}\leqslant v\leqslant 8-6u.\end{cases}$$

This gives

$$\big(P(u,v)\big)^2=\begin{cases}27u^2-v^2-72u+48 \text{ if } 0\leqslant v\leqslant \dfrac{12-9u}{2},\\ 3(8-6u-v)^2 \text{ if } \dfrac{12-9u}{2}\leqslant v\leqslant 8-6u,\end{cases}$$

and

$$P(u,v)\cdot F=\begin{cases}v \text{ if } 0\leqslant v\leqslant \dfrac{12-9u}{2},\\ 24-18u-3v \text{ if } \dfrac{12-9u}{2}\leqslant v\leqslant 8-6u.\end{cases}$$

Thus, if $P \in E'$, then

$$S\big(W^S_{\bullet,\bullet};F\big) = \frac{3}{20}\int_1^{\frac{4}{3}}(27u^2-72u+48)(u-1)du + \frac{3}{20}\int_0^1\int_0^{\frac{6-3u}{2}} u^2-v^2-8u+10dvdu+$$
$$+\frac{3}{20}\int_0^1\int_{\frac{6-3u}{2}}^{3-u} 10u^2+12uv+3v^2-44u-24v+46dvdu+\frac{3}{20}\int_0^1\int_{3-u}^{\frac{10-4u}{3}}(4u+3v-10)^2dvdu+$$
$$+\frac{3}{20}\int_1^{\frac{4}{3}}\int_0^{\frac{12-9u}{2}} 27u^2-v^2-72u+48dvdu+\frac{3}{20}\int_1^{\frac{4}{3}}\int_{\frac{12-9u}{2}}^{8-6u}3(6u+v-8)^2dvdu=\frac{41}{24}<2.$$

Similarly, if $P \notin E'$, then $S(W^S_{\bullet,\bullet};F) = \frac{409}{240} < \frac{41}{24} < 2$.

Now, let O be a point in F. Let us compute $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)$. We have

$$S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O) = \frac{3}{20}\int_0^1\int_0^{\frac{6-3u}{2}} v^2dvdu+\frac{3}{20}\int_0^1\int_{\frac{6-3u}{2}}^{3-u}(12-6u-3v)^2dvdu+$$
$$+\frac{3}{20}\int_0^1\int_{3-u}^{\frac{10-4u}{3}}(30-12u-9v)^2dvdu+\frac{3}{20}\int_1^{\frac{4}{3}}\int_0^{\frac{12-9u}{2}}8v^2dvdu+\frac{3}{20}\int_1^{\frac{4}{3}}\int_{\frac{12-9u}{2}}^{8-6u}(24-18u-3v)^2dvdu+F_O\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}\big),$$

so that $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O) = \frac{63}{80} + F_O(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet})$. In particular, if $P \notin E'$ and $O \notin \widetilde{T} \cup \widetilde{C}$, then $F_O(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}) = 0$, which implies that $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O) = \frac{63}{80}$. Let us compute $F_O(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet})$ in the remaining cases.

First, we deal with the case $P \notin E'$. If $P \notin E'$, then we have $O \notin \mathrm{Supp}(\widetilde{N}'(u))$ for every $u \in [0, \frac{4}{3}]$. Moreover, if $P \notin E'$ and $O \in \widetilde{L}$, then $O \notin \widetilde{T}$, and $\widetilde{L}$ intersects F transversally at O, which gives

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big) = \frac{63}{80} + \frac{6}{20}\int_0^1\int_{3-u}^{\frac{10-4u}{3}}\big(P(u,v)\cdot F\big)(v+u-3)\big(\widetilde{L}\cdot F\big)_O dvdu = \frac{19}{24}.$$

Similarly, if $P \notin E'$ and $O \in \widetilde{T}$, then $O \notin \widetilde{L}$ and

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big) = \frac{63}{80} + \frac{6}{20}\int\limits_0^1 \int\limits_{\frac{6-3u}{2}}^{\frac{10-4u}{3}} \big(P(u,v)\cdot F\big)(2v-6+3u)\big(\widetilde{T}\cdot F\big)_O dvdu +$$

$$+\frac{6}{20}\int\limits_1^{\frac{4}{3}} \int\limits_{\frac{12-9u}{2}}^{8-6u} \big(P(u,v)\cdot F\big)(2v+9u-12)\big(\widetilde{T}\cdot F\big)_O = \frac{63}{80} + \frac{6}{20}\int\limits_0^1 \int\limits_{\frac{6-3u}{2}}^{3-u} (12-6u-3v)(2v-6+3u)\big(\widetilde{T}\cdot F\big)_O dvdu +$$

$$+\frac{6}{20}\int\limits_0^1 \int\limits_{3-u}^{\frac{10-4u}{3}} (30-12u-9v)(2v-6+3u)\big(\widetilde{T}\cdot F\big)_O dvdu + \frac{6}{20}\int\limits_1^{\frac{4}{3}} \int\limits_{\frac{12-9u}{2}}^{8-6u} (24-18u-3)(2v+9u-12)\big(\widetilde{T}\cdot F\big)_O,$$

so $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O) = \frac{63}{80} + \frac{5}{96}\big(\widetilde{T}\cdot F\big)_O \leqslant \frac{63}{80} + \frac{5}{96}\widetilde{T}\cdot F = \frac{107}{120}$. Hence, if $P \notin E'$, then $\beta(\mathbf{F}) > 0$ by (8).

Therefore, to complete the proof of the lemma, we may assume that $P \in E'$. Since R is smooth, the curve $\widetilde{R}$ intersects F transversally at one point, so that

$$\mathrm{ord}_O\big(\widetilde{N}'(u)\big|_F\big) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ 0 \text{ if } 1 \leqslant u \leqslant \dfrac{4}{3} \text{ and } O \neq \widetilde{R}\cap F, \\ u-1 \text{ if } 1 \leqslant u \leqslant \dfrac{4}{3} \text{ and } O = \widetilde{R}\cap F. \end{cases}$$

Hence, if $O \neq \widetilde{R}\cap F$, then $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)$ can be computed as in the case $P \notin E'$. Thus, we may also assume that $O = \widetilde{R}\cap F$. Moreover, if $O \in \widetilde{L}$, then our previous calculations give

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big) = \frac{6}{20}\int\limits_1^{\frac{4}{3}} \int\limits_0^{\widetilde{t}(u)} \big(\widetilde{P}(u,v)\cdot F\big)(u-1)dvdu + \frac{19}{24} =$$

$$= \frac{6}{20}\int\limits_1^{\frac{4}{3}} \int\limits_0^{\frac{12-9u}{2}} v(u-1)dvdu + \frac{6}{20}\int\limits_1^{\frac{4}{3}} \int\limits_{\frac{12-9u}{2}}^{8-6u} (24-18u-3v)(u-1)dvdu + \frac{19}{24} = \frac{191}{240}.$$

Similarly, if $O \in \widetilde{T}$, then, using our previous computations, we get

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big) = \frac{1}{241} + \frac{63}{80} + \frac{5}{96}\big(\widetilde{T}\cdot F\big)_O \leqslant \frac{1}{241} + \frac{63}{80} + \frac{5}{96}\widetilde{T}\cdot F = \frac{43}{48}.$$

Thus, we see that $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O) < 1$ for every point $O \in F$, so that $\beta(\mathbf{F}) > 0$ by (8). □

To complete the proof of Theorem A, we may assume that $T=\ell+C_2$ and $P\in\ell\cap C_2$, where ℓ is a line such that $\pi(\ell)$ is a conic in $\mathbb{P}^2$, and C_2 is a smooth conic such that $\pi(C_2)$ is a line. Then C_2 is one of the curves ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5, ℓ_6, so we may assume that $C_2=\ell_6$. Set $L'=\ell_1+\ell_2+\ell_3+\ell_4+\ell_5$. Let us denote by $\widetilde{\ell}$, $\widetilde{C}_2$, $\widetilde{L}'$ the strict transforms on the surface $\widetilde{S}$ of the curves ℓ, C_2, L', respectively. Then $\widetilde{\ell}\cap\widetilde{L}'=\varnothing$ and $\widetilde{C}_2\cap\widetilde{L}'=\varnothing$. Moreover, if $0\leqslant u\leqslant 1$, then by considering intersection with the curves $\widetilde{\ell},\widetilde{C_2}$ and $\widetilde{L}'$, we obtain:

$$P(u,v)\sim_{\mathbb{R}}\begin{cases}\frac{2+u}{3}\widetilde{\ell}+\widetilde{C}_2+\frac{1-u}{3}\widetilde{L}'+\frac{10-4u-3v}{3}F \text{ if } 0\leqslant v\leqslant 3-2u,\\ \frac{13-4u-3v}{6}\widetilde{\ell}+\widetilde{C}_2+\frac{1-u}{3}\widetilde{L}'+\frac{10-4u-3v}{3}F \text{ if } 3-2u\leqslant v\leqslant \frac{9-4u}{3},\\ \frac{10-4u-3v}{3}\big(2\widetilde{\ell}+3\widetilde{C}_2+F\big)+\frac{1-u}{3}\widetilde{L}' \text{ if } \frac{9-4u}{3}\leqslant v\leqslant 3-u,\\ \frac{10-4u-3v}{3}\big(2\widetilde{\ell}+\widetilde{L}'+3\widetilde{C}_2+F\big) \text{ if } 3-u\leqslant v\leqslant \frac{10-4u}{3},\end{cases}$$

and

$$N(u,v)=\begin{cases}0 \text{ if } 0\leqslant v\leqslant 3-2u,\\ \frac{v+2u-3}{2}\widetilde{\ell} \text{ if } 3-2u\leqslant v\leqslant \frac{9-4u}{3},\\ (2v+3u-6)\widetilde{\ell}+(3v+4u-9)\widetilde{C}_2 \text{ if } \frac{9-4u}{3}\leqslant v\leqslant 3-u,\\ (2v+3u-6)\widetilde{\ell}+(3v+4u-9)\widetilde{C}_2+(v+u-3)\widetilde{L}' \text{ if } 3-u\leqslant v\leqslant \frac{10-4u}{3}.\end{cases}$$

This gives

$$\big(P(u,v)\big)^2=\begin{cases}u^2-v^2-8u+10 \text{ if } 0\leqslant v\leqslant 3-2u,\\ \frac{29}{2}-14u-3v+3u^2-\frac{v^2}{2}+2vu \text{ if } 3-2u\leqslant v\leqslant \frac{9-4u}{3},\\ 11u^2+14uv+4v^2-50u-30v+55 \text{ if } \frac{9-4u}{3}\leqslant v\leqslant 3-u,\\ (10-4u-3v)^2 \text{ if } 3-u\leqslant v\leqslant \frac{10-4u}{3},\end{cases}$$

and

$$P(u,v)\cdot F=\begin{cases}v \text{ if } 0\leqslant v\leqslant 3-2u,\\ \frac{3}{2}-u+\frac{v}{2} \text{ if } 3-2u\leqslant v\leqslant \frac{9-4u}{3},\\ 15-7u-4v \text{ if } \frac{9-4u}{3}\leqslant v\leqslant 3-u,\\ 30-12u-9v \text{ if } 3-u\leqslant v\leqslant \frac{10-4u}{3}.\end{cases}$$

Furthermore, if $1 \leqslant u \leqslant \frac{4}{3}$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-3u)(\widetilde{\ell}+\widetilde{C}_2)+(8-6u-v)F \text{ if } 0 \leqslant v \leqslant 4-3u, \\ \dfrac{12-9u-v}{2}\widetilde{\ell}+(4-3u)\widetilde{C}_2+(8-6u-v)F \text{ if } 4-3u \leqslant v \leqslant \dfrac{20-15u}{3}, \\ (8-6u-v)\big(2\widetilde{\ell}+3\widetilde{C}_2+F\big) \text{ if } \dfrac{20-15u}{3} \leqslant v \leqslant 8-6u, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant 4-3u, \\ \dfrac{v+3u-4}{2}\widetilde{\ell} \text{ if } 4-3u \leqslant v \leqslant \dfrac{20-15u}{3}, \\ (9u+2v-12)\widetilde{\ell}+(15u+3v-20)\widetilde{C}_2 \text{ if } \dfrac{20-15u}{3} \leqslant v \leqslant 8-6u. \end{cases}$$

This gives

$$\big(P(u,v)\big)^2 = \begin{cases} 27u^2-v^2-72u+48 \text{ if } 0 \leqslant v \leqslant 4-3u, \\ 56-84u-4v+\dfrac{63}{2}u^2-\dfrac{v^2}{2}+3vu \text{ if } 4-3u \leqslant v \leqslant \dfrac{20-15u}{3}, \\ 4(8-6u-v)^2 \text{ if } \dfrac{20-15u}{3} \leqslant v \leqslant 8-6u, \end{cases}$$

and

$$P(u,v)\cdot F = \begin{cases} v \text{ if } 0 \leqslant v \leqslant 4-3u, \\ 2-\dfrac{3u}{2}+\dfrac{v}{2} \text{ if } 4-3u \leqslant v \leqslant \dfrac{20-15u}{3}, \\ 32-24u-4v \text{ if } \dfrac{20-15u}{3} \leqslant v \leqslant 8-6u. \end{cases}$$

Now, as in the proof of Lemma 6.3.5, we compute

$$S\big(W^S_{\bullet,\bullet};F\big) = \begin{cases} \dfrac{77}{45} \text{ if } P \in E', \\ \dfrac{1229}{720} \text{ if } P \notin E'. \end{cases}$$

Similarly, if O is a point in F, we can compute $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)$ as we did in the proof of Lemma 6.3.5. The results of these computations are presented in the following two tables:

condition	$O \in \widetilde{\ell}\cap\widetilde{C}_2\cap\widetilde{R}$	$\widetilde{\ell}\cap\widetilde{C}_2 \ni O \notin \widetilde{R}$	$\widetilde{\ell}\cap\widetilde{R} \ni O \notin \widetilde{C}_2$	$\widetilde{\ell} \ni O \notin \widetilde{R}\cup\widetilde{C}_2$	$\widetilde{C}_2\cap\widetilde{R} \ni O \notin \widetilde{\ell}$
$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)$	$\frac{163}{180}$	$\frac{649}{720}$	$\frac{1859}{2160}$	$\frac{185}{216}$	$\frac{1801}{2160}$

condition	$\widetilde{C}_2 \ni O \notin \widetilde{R}\cup\widetilde{\ell}$	$O \in \widetilde{L}'\cap\widetilde{R}$	$\widetilde{L}' \ni O \notin \widetilde{R}$	$\widetilde{R} \ni O \notin \widetilde{\ell}\cup\widetilde{C}'\cap\widetilde{L}'$	$O \notin \widetilde{\ell}\cup\widetilde{C}'\cap\widetilde{L}'\cup\widetilde{R}$
$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)$	$\frac{112}{135}$	$\frac{571}{720}$	$\frac{71}{90}$	$\frac{71}{90}$	$\frac{113}{144}$

Thus, we proved that $S(W^S_{\bullet,\bullet};F)<2$, and we proved that $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)<1$ for every point $O\in F$. Therefore, using (8), we get $\beta(\mathbf{F})>0$. This completes the proof of Theorem A.

6.4 The proof of Theorem B

Let us use all assumptions and notations introduced in Section 6.1. Recall that

$$\mathrm{Aut}\big(\mathbb{P}^3,C_6\big)\simeq \mathrm{Aut}\big(C,[D]\big)\subset \mathrm{Aut}(C).$$

For a general quartic curve C, the automorphism group is trivial. The cases for which it is not are listed in Bars Cortina (2006); Dolgachev (2012). We recall them here.

Proposition 6.4.1 ((Bars Cortina, 2006, Theorem 16), (Dolgachev, 2012, Table 6)). *Let C be a smooth plane quartic curve such that the automorphism group $\mathrm{Aut}(C)$ is not trivial. Then the possibilities for C and $\mathrm{Aut}(C)$ are as follows.*

$\mathrm{Aut}(C)$	GAP ID	Equation of the curve C	Notes
$\mathbb{Z}_2$	[2,1]	$z^4+\lambda z^2P(x,y)+Q(x,y)=0$	$\deg(P)=2,\deg(Q)=4$, C not one of the curves below
$\mathbb{Z}_3$	[3,1]	$z^3P(x,y)+Q(x,y)=0$	$\deg(P)=1,\deg(Q)=4$, C not one of the curves below
$\mathbb{Z}_2\times\mathbb{Z}_2$	[4,2]	$x^4+y^4+z^4+z^2(\lambda x^2+\mu y^2)+\gamma x^2y^2=0$	$\lambda\neq\gamma,\mu\neq\gamma,\lambda\neq\mu$
$\mathfrak{S}_3$	[6,1]	$z^4+z(x^3+y^3)+\lambda z^2xy+\mu x^2y^2=0$	$\lambda\neq\mu,\lambda\mu\neq 0$
$\mathbb{Z}_6$	[6,2]	$z^4+\lambda y^2z^2+y^4+x^3z=0$	$\lambda\neq 0$
D_4	[8,3]	$x^4+y^4+z^4+\lambda z^2(x^2+y^2)+\mu x^2y^2=0$	$\lambda\neq 0$ and $\lambda\neq\mu$
$\mathbb{Z}_9$	[9,1]	$z^4+zy^3+yx^3=0$	
$\mathrm{D}_4.\mathbb{Z}_2$	[16,13]	$x^4+y^4+z^4+\lambda x^2y^2=0$	$\lambda\notin\{0,\pm2,\pm6,\pm2i\sqrt{3}\}$
$\mathfrak{S}_4$	[24,12]	$x^4+y^4+z^4+\lambda(x^2y^2+x^2z^2+y^2z^2)=0$	$\lambda\notin\{-1,2,-2,0,\frac{-3\pm3i\sqrt{7}}{2}\}$
$\mathbb{Z}_4.\mathfrak{A}_4$	[48,33]	$y^4-x^3z+z^4=0$	
$\mathbb{Z}_4^2\rtimes\mathfrak{S}_3$	[96,64]	$x^4+y^4+z^4=0$	
$\mathrm{PSL}_2(\mathbb{F}_7)$	[168, 42]	$x^3y+y^3z+z^3x=0$	

Since π is $\mathrm{Aut}(\mathbb{P}^3, C_6)$-equivariant, we can also identify $\mathrm{Aut}(\mathbb{P}^3, C_6)$ with a subgroup in $\mathrm{Aut}(X)$. Then the action of the group $\mathrm{Aut}(X)$ on the set $\{E, E'\}$ gives a monomorphism

$$\mathrm{Aut}(X)/\mathrm{Aut}\big(\mathbb{P}^3, C_6\big) \hookrightarrow \mathbb{Z}_2,$$

which is surjective if and only if $\mathrm{Aut}(X)$ has an element that swaps the surfaces E and E'.

Remark 6.4.2 ((Dolgachev, 2012, Example 7.2.6))**.** We can choose M_1, M_2, M_3 in (2) to be symmetric $\iff 2D \sim K_C$. Moreover, if M_1, M_2, M_3 are symmetric, then X admits the involution

$$\big([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]\big) \mapsto \big([y_0 : y_1 : y_2 : y_3], [x_0 : x_1 : x_2 : x_3]\big).$$

In this case, we have $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{P}^3, C_6) \times \mathbb{Z}_2$. For more details, see Ottaviani (2024).

Remark 6.4.3 (Kuznetsov)**.** Set $V = H^0(\mathscr{O}_C(K_C + D))$, $W = H^0(\mathscr{O}_C(2K_C - D))$ and $G = \mathrm{Aut}(C, [D])$. Let $\widehat{G}$ be a central extension of the group G such that D (considered as a line bundle) is $\widehat{G}$-linearizable. Then the sheaf $\mathscr{O}_C(D)$ admits a $\widehat{G}$-equivariant resolvent

$$0 \to W^* \otimes \mathscr{O}_{\mathbb{P}^2}(-2) \to V \otimes \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_C(D) \to 0,$$

which is known as the Beilinson resolvent. Since $W^* \otimes \mathscr{O}_{\mathbb{P}^2}(-2) \to V \otimes \mathscr{O}_{\mathbb{P}^2}(-1)$ is $\widehat{G}$-equivariant, the corresponding map $\rho : V^* \otimes W^* \to H^0\big(\mathscr{O}_{\mathbb{P}^2}(1)\big)$ is equivariant, where $H^0\big(\mathscr{O}_{\mathbb{P}^2}(1)\big) \simeq H^0\big(\mathscr{O}_C(K_C)\big)$ as $\widehat{G}$-representations. On the other hand, the embedding $X \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^3$ given by (2) can be realized as

$$X = \big(\mathbb{P}(V^*) \times \mathbb{P}(W^*)\big) \cap \mathbb{P}\big(\ker(\rho)\big),$$

and the $\widehat{G}$-action on X factors through G, which is the natural G-action.

This remark gives the following.

Lemma 6.4.4. *There exists a group homomorphism* $\eta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(C)$ *such that its restriction to the subgroup* $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathrm{Aut}(C, [D])$ *gives a natural embedding* $\mathrm{Aut}(\mathbb{P}^3, C_6) \hookrightarrow \mathrm{Aut}(C)$.

Proof. Let $\mathscr{M}$ be the two-dimensional linear system of divisors of degree $(1,1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$ that contains the threefold X. Then $\mathscr{M}$ can be identified with the projectivization of the three-dimensional vector space spanned by the matrices M_1, M_2, M_3, which we will identify with $\mathbb{P}^2_{x,y,z}$. Then $\mathrm{Aut}(X)$ naturally acts on this $\mathbb{P}^2_{x,y,z}$, because the action of the group $\mathrm{Aut}(X)$ on X lifts to its action on $\mathbb{P}^3 \times \mathbb{P}^3$.

Moreover, the $\mathrm{Aut}(X)$-action on $\mathbb{P}^2_{x,y,z}$ preserves the quartic curve in $\mathbb{P}^2_{x,y,z}$ given by

$$\det\big(xM_1 + yM_2 + zM_2\big) = 0,$$

which parametrizes singular divisors in $\mathscr{M}$. This curve is isomorphic to the curve C, which gives us the required homomorphism of groups $\eta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(C)$. It follows from Remark 6.4.3 that this group homomorphism is functorial, so it gives a natural embedding $\mathrm{Aut}(\mathbb{P}^3, C_6) \hookrightarrow \mathrm{Aut}(C)$. □

Corollary 6.4.5. *Either* $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{P}^3, C_6) \times \mathbb{Z}_2$ *or* $\mathrm{Aut}(X)$ *is isomorphic to a subgroup* $\mathrm{Aut}(C)$.

Now, we are ready to state a criterion when $\mathrm{Aut}(X) \neq \mathrm{Aut}(\mathbb{P}^3, C_6)$.

Lemma 6.4.6. $\mathrm{Aut}(X) \neq \mathrm{Aut}(\mathbb{P}^3, C_6) \iff$ *there is* $h \in \mathrm{Aut}(C)$ *such that* $h^*(D) \sim K_C - D$.

Proof. By Remark 6.4.3, the left copy of $\mathbb{P}^3$ in (3) can be be identified with $\mathbb{P}(H^0(\mathscr{O}_C(K_C + D))^\vee)$, while the right copy of $\mathbb{P}^3$ can be be identified with $\mathbb{P}(H^0(\mathscr{O}_C(2K_C - D))^\vee)$. Thus, if $\mathrm{Aut}(C)$ contains an automorphism h such that $h^*(D) \sim K_C - D$, we can use it to identify both copies of $\mathbb{P}^3$ in (3), which will give us an automorphism of X that swaps exceptional surfaces of the blow ups π and π'.

Vice versa, if the group $\mathrm{Aut}(X)$ is larger than $\mathrm{Aut}(\mathbb{P}^3, C_6)$, it follows from the proof of Lemma 6.4.4 that there exists $h \in \mathrm{Aut}(C)$ such that $h^*(D) \sim K_C - D$. □

Recall that $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathrm{Aut}(C, [D])$, where D is a divisor on C of degree 2 that satisfies (1). Using Remark 6.4.2, Lemma 6.4.6 and its proof, we obtain

Corollary 6.4.7. *One of the following three cases holds:*

- $2D \sim K_C$ *and* $\mathrm{Aut}(X) \simeq \mathrm{Aut}(C, [D]) \times \mathbb{Z}_2$,
- $2D \not\sim K_C$, *there is* $h \in \mathrm{Aut}(C)$ *such that* $h^*(D) \sim K_C - D$, *and*

$$\mathrm{Aut}(X) \simeq \big\langle \mathrm{Aut}(C, [D]), h \big\rangle.$$

- $\mathrm{Aut}(X) \simeq \mathrm{Aut}(C, [D])$, *and* $h^*(D) \not\sim K_C - D$ *for every* $h \in \mathrm{Aut}(C)$.

Corollary 6.4.8. *If* $\mathrm{Aut}(X)$ *is not isomorphic to any subgroup of* $\mathrm{Aut}(C)$, *then* $2D \sim K_C$.

Using Corollary 6.4.7, we can find all possibilities for $\mathrm{Aut}(X)$, but this requires a lot of work, because we have to analyze $\mathrm{Pic}^G(C)$ for every subgroup $G \subset \mathrm{Aut}(C)$. This can be done using the following proposition and remark.

Proposition 6.4.9 ((Dolgachev, 1999, Proposition 2.2))**.** *Let G be a subgroup in* $\mathrm{Aut}(C)$. *Then there exists exact sequence*

$$1 \to \mathrm{Hom}\big(G, \mathbb{C}^*\big) \to \mathrm{Pic}\big(G, C\big) \to \mathrm{Pic}^G\big(C\big) \to H^2\big(G, \mathbb{C}^*\big) \to 1,$$

where $\mathrm{Pic}(G, C)$ *is the group of G-linearized line bundles on C modulo G-equivariant isomorphisms.*

Remark 6.4.10. Let G be a subgroup in $\mathrm{Aut}(C)$, let $\Sigma_1,\ldots,\Sigma_n$ be all G-orbits in C of length less that $|G|$. We may assume that $|\Sigma_i| \geqslant |\Sigma_j|$ for $i \geqslant j$. For every $i \in \{1,\ldots,n\}$, set

$$e_i = \frac{|G|}{|\Sigma_i|} = \text{the order of the stabilizer in } G \text{ of a point in } \Sigma_i.$$

The *signature* of the G-action on C is the tuple $\big[g;e_1,\ldots,e_n\big]$, where g is the genus of the curve C/G. If $C/G \simeq \mathbb{P}^1$, then it follows from (Dolgachev, 1999, (2.2)) that

$$\mathrm{Pic}\big(G,C\big) \simeq \mathbb{Z}\oplus\mathbb{Z}_{a_1}\oplus\mathbb{Z}_{a_2}\oplus\cdots\oplus\mathbb{Z}_{a_{n-1}}$$

for $a_1 = d_1, a_2 = \frac{d_2}{d_1},\ldots,a_{n-1} = \frac{d_{n-1}}{d_{n-2}}$, where

$$\begin{aligned} d_1 &= \gcd(e_1,\ldots,e_n),\\ d_2 &= \gcd(e_1e_2,e_1e_3,\ldots,e_ie_j,\ldots,e_{n-1}e_n),\\ &\vdots\\ d_{n-1} &= \gcd(e_1e_2\cdots e_{n-1},\ldots,e_2\cdots e_{n-1}e_n). \end{aligned}$$

Moreover, if γ is a generator of the free part of $\mathrm{Pic}^G(C)$ in this case, then it follows from it follows from (Dolgachev, 1999, (2.3)) that

$$4 = \deg(K_C) = \mathrm{lcm}(e_1,\ldots,e_n)\left(n-2-\sum_{i=1}^{n}\frac{1}{e_i}\right)\deg(\gamma).$$

Let us show how to compute $\mathrm{Pic}^G(C)$ in some cases.

Example 49. Suppose that $\mathrm{Aut}(C)$ has a subgroup $G \simeq \mathfrak{S}_4$. Then, by Proposition 6.4.1, the quartic curve C can be given in $\mathbb{P}^2_{x,y,z}$ by

$$x^4+y^4+z^4+\lambda(x^2y^2+x^2z^2+y^2z^2) = 0$$

for some $\lambda \in \mathbb{C}$ such that $\lambda \notin \{-1,2,-2\}$. One can show that

$$\mathrm{Aut}(C) \simeq \begin{cases} \mathfrak{S}_4 \text{ if } \lambda \neq 0 \text{ and } \lambda \neq \dfrac{-3\pm3\sqrt{7}i}{2},\\ \mathbb{Z}_4^2 \rtimes \mathfrak{S}_3 \text{ if } \lambda = 0,\\ \mathrm{PSL}_2(\mathbb{F}_7) \text{ if } \lambda = \dfrac{-3\pm3\sqrt{7}i}{2}. \end{cases}$$

We have $C/G \simeq \mathbb{P}^1$, and it follows from LMFDB Collaboration (2025) that the signature is $[0;2,2,2,3]$. Thus, using Remark 6.4.10, we see that $\mathrm{Pic}(G,C) \simeq \mathbb{Z}\times\mathbb{Z}_2^2$, and the free part of the group $\mathrm{Pic}(G,C)$ is generated by K_C. Moreover, using GAP, we compute $\mathrm{Hom}(G,\mathbb{C}^*) \simeq H^2(G,\mathbb{C}^*) \simeq \mathbb{Z}_2$. Therefore, using Proposition 6.4.9, we get the following exact sequence of group homomorphisms:

$$0 \to \mathbb{Z} \times \mathbb{Z}_2 \to \mathrm{Pic}^G(C) \to \mathbb{Z}_2 \to 0.$$

We also know from Disney-Hogg (2024) that $\mathrm{Pic}(C)$ contains two G-invariant even theta-characteristics θ_1 and θ_2. This immediately implies that $\mathrm{Pic}^G(C) = \langle \theta_1, \theta_2 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_2$.

Example 50 (Dolgachev (1999))**.** Suppose that $\mathrm{Aut}(C) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$. Then C is given in $\mathbb{P}^2_{x,y,z}$ by

$$xy^3 + yz^3 + zx^3 = 0.$$

Set $G = \mathrm{Aut}(C)$. Using Example 44, we conclude that $\mathrm{Pic}^G(C)$ contains an even theta-characteristic θ. Now, arguing as in Example 49, we get $\mathrm{Pic}^G(C) = \langle \theta \rangle \simeq \mathbb{Z}$.

Example 51. Suppose that $\mathrm{Aut}(C) \simeq \mathbb{Z}_4^2 \rtimes \mathfrak{S}_3$, of GAP ID [96,64]. Then C is given in $\mathbb{P}^2_{x,y,z}$ by

$$x^4 + y^4 + z^4 = 0,$$

the group $\mathrm{Aut}(C)$ contains a unique subgroup isomorphic to $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$, and C is the unique plane quartic curve admiting a faithful $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$-action. Let G be this subgroup. Then the signature is $[0;3,3,4]$. Therefore, using Remark 6.4.10, we get $\mathrm{Pic}(G,C) \simeq \mathbb{Z} \times \mathbb{Z}_3$, where the free part is generated by K_C. Since $\mathrm{Hom}(G,\mathbb{C}^*) \simeq \mathbb{Z}_3$ and $H^2(G,\mathbb{C}^*) \simeq \mathbb{Z}_4$, it follows from Proposition 6.4.9 that

$$\mathrm{Pic}^G(C)/\langle K_C \rangle \simeq \mathbb{Z}_4.$$

Moreover, we know from Section 6.2.2 that $\mathrm{Pic}^G(C)$ contains a divisor D of degree 2. Thus, we conclude that $\mathrm{Pic}^G(C) = \langle K_C, D \rangle \simeq \mathbb{Z} \times \mathbb{Z}_2$, and $K_C - 2D$ is a two-torsion divisor.

Example 52. Let C be the Fermat quartic curve from Example 51, and let $G = \mathrm{Aut}(C) \simeq \mathbb{Z}_4^2 \rtimes \mathfrak{S}_3$. Then the signature is $[0;2,3,8]$, so it follows from Remark 6.4.10 that

$$\mathrm{Pic}(G,C) \simeq \mathbb{Z} \times \mathbb{Z}_2,$$

where the free part is generated by K_C. One can check that $\mathrm{Hom}(G,\mathbb{C}^*) \simeq \mathbb{Z}_2$ and $H^2(G,\mathbb{C}^*) \simeq \mathbb{Z}_2$. We claim that $\mathrm{Pic}^G(C)$ has no divisors of degree 2. Indeed, if $\mathrm{Pic}^G(C)$ has a divisor D of degree 2, then $|K_C + D|$ gives a G-equivariant embedding $C \hookrightarrow \mathbb{P}^3$, and we can identify C with its image in $\mathbb{P}^3$. Then $\mathrm{Aut}(\mathbb{P}^3, C) \simeq G$, and $\mathbb{P}^3$ has no $\mathrm{Aut}(\mathbb{P}^3, C)$-orbits of length 1 or 2. This contradicts Lemma 6.2.6, because $\mathbb{Z}_2^3.\mathfrak{S}_4$ does not contain subgroups isomorphic to G. Therefore, we see that $\mathrm{Pic}^G(C)$ does not contain divisors of degree 2, so arguing as in Example 51, we get

$$\mathrm{Pic}^G(C) = \langle K_C, \delta \rangle \simeq \mathbb{Z} \times \mathbb{Z}_2,$$

where δ is a two-torsion divisor.

Using results described in Examples 49, 50, 51, 52, we get the following corollaries:

Corollary 6.4.11. *If* $\mathrm{Aut}(\mathbb{P}^3, C_6)$ *has a subgroup isomorphic to* $\mathfrak{S}_4$*, then one of the following holds:*

- $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathfrak{S}_4$ *and* $\mathrm{Aut}(X) \simeq \mathfrak{S}_4 \times \mathbb{Z}_2$*,*
- $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$ *and* $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}_2$*.*

Corollary 6.4.12. *The smooth Fano threefold described in Example 44 is the unique smooth Fano threefold in the deformation family №2.12 that admits a faithful action of the group* $\mathrm{PSL}_2(\mathbb{F}_7)$*.*

Corollary 6.4.13. *The smooth Fano threefold described in Section 6.2.2 is the only smooth Fano threefold in the family №2.12 that admits a faithful action of the group* $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$*.*

Proof. Suppose that $\mathrm{Aut}(X)$ has a subgroup isomorphic to $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$. Then arguing as in Example 52, we see that $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$, and $\mathrm{Aut}(\mathbb{P}^3, C_6)$ is conjugate to the subgroup G that has been described in Section 6.2.2. Thus, the required assertion follows from Theorem 6.2.9. □

Now, we are ready to prove Theorem B.

Proof of Theorem B. It is enough to show that the automorphism group $\mathrm{Aut}(X)$ is isomorphic to a subgroup of $\mathrm{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}_2$ or $\mathbb{Z}_4^2 \rtimes \mathfrak{S}_3$. Suppose this is not true. Let us seek for a contradiction.

Using a variation of the Magma code used in Braden and Disney-Hogg (2025); Disney-Hogg (2024) provided to us by Linden Disney-Hogg, we compute the lengths of K-orbits of theta-characteristics for every possible subgroup $K \subseteq \mathrm{Aut}(C)$. This shows that if a subgroup in $\mathrm{Aut}(C)$ leaves an even theta-characteristic invariant, then this subgroup is isomorphic to a subgroup of $\mathrm{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}_2$ or $\mathbb{Z}_4^2 \rtimes \mathfrak{S}_3$.

Let $G = \mathrm{Aut}(C, [D])$. Since G is not contained in $\mathrm{PSL}_2(\mathbb{F}_7) \times \mathbb{Z}_2$ or $\mathbb{Z}_4^2 \rtimes \mathfrak{S}_3$, the curve C does not contain G-invariant even theta-characteristics. Therefore, D is not an even theta characteristic. Thus, by Corollary 6.4.8, the group $\mathrm{Aut}(X)$ is isomorphic to a subgroup of the group $\mathrm{Aut}(C)$.

Now, using Proposition 6.4.1, we find that the only possibilities of $\mathrm{Aut}(X)$ are:

$\mathbb{Z}_9$, $\mathbb{Z}_{12}$, $\mathrm{SL}_2(\mathbb{F}_3)$ (GAP ID is [24,3]), $\mathbb{Z}_4.\mathfrak{A}_4$ (GAP ID is [48,33]),

Moreover, by Corollary 6.4.7, either $G = \mathrm{Aut}(X)$ or G is a subgroup in $\mathrm{Aut}(X)$ of index 2. Thus, we have the following possibilities:

$\mathrm{Aut}(X)$	$\mathbb{Z}_9$	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathrm{SL}_2(\mathbb{F}_3)$	$\mathbb{Z}_4.\mathfrak{A}_4$	$\mathbb{Z}_4.\mathfrak{A}_4$
G	$\mathbb{Z}_9$	$\mathbb{Z}_6$	$\mathbb{Z}_{12}$	$\mathrm{SL}_2(\mathbb{F}_3)$	$\mathrm{SL}_2(\mathbb{F}_3)$	$\mathbb{Z}_4.\mathfrak{A}_4$

Recall that D is a divisor on the quartic curve C such that $\deg(D) = 2$, the divisor D satisfies (1), and its class $[D] \in \mathrm{Pic}(C)$ is G-invariant. Let us show that in each of our cases, such D does not exist.

For each G, Proposition 6.4.1 lists all possibilities for the equation of C, and using Remark 6.4.10, we can describe the signature of the G-action on C, as well as the degree of a generator γ of the free part of the group $\mathrm{Pic}^G(C)$. Finally, Proposition 6.4.9 allows us to compute the structure of the group $\mathrm{Pic}^G(S)$. This gives the following possibilities:

G	Equation of C	Signature	Structure of $\mathrm{Pic}^G(S)$	$\deg(\gamma)$
$\mathbb{Z}_6$	$y^4 - x^3z + z^4 + \lambda y^2z^2 = 0,\ \lambda \in \mathbb{C}$	$[0;2,3,3,6]$	$\mathbb{Z} \times \mathbb{Z}_3$	1
$\mathbb{Z}_9$	$y^3z - x(x^3 - z^3) = 0$	$[0;3,9,9]$	$\mathbb{Z} \times \mathbb{Z}_3$	1
$\mathbb{Z}_{12}$	$y^4 - x^3z + z^4 = 0$	$[0;3,4,12]$	$\mathbb{Z}$	1
$\mathrm{SL}_2(3)$	$y^4 - x^3z + z^4 = 0$	$[0;2,3,6]$	$\mathbb{Z} \times \mathbb{Z}_6$	4
$\mathbb{Z}_4.\mathfrak{A}_4$	$y^4 - x^3z + z^4 = 0$	$[0;2,3,12]$	$\mathbb{Z}$	4

In particular, if $G \simeq \mathrm{SL}_2(3)$ or $G \simeq \mathbb{Z}_4.\mathfrak{A}_4$, then C does not have G-invariant divisors of degree 2. Hence, we see that G is isomorphic to one of the following groups: $\mathbb{Z}_6$, $\mathbb{Z}_9$, $\mathbb{Z}_{12}$.

Suppose that $G \simeq \mathbb{Z}_{12}$. Then the action of G on C is generated by

$$[x:y:z] \mapsto \big[\omega_3 x : iy : z\big],$$

where ω_3 is a primitive cube root of the unity. Then G fixes the point $P = [1:0:0]$, which implies that $\mathrm{Pic}^G(S) = \mathbb{Z}[P]$, so that $D \sim 2P$, which contradicts to our assumption that D satisfies (1).

Assume now that $G \simeq \mathbb{Z}_9$. Then the G-action on the curve is given by

$$[x:y:z] \mapsto \big[\omega_9 x : \omega_9^{-3} y : z\big],$$

where ω_9 is a primitive ninth root of the unity. Set $P_1 = [0:1:0]$ and $P_2 = [0:0:1]$. Then

$$\{z = 0\} \cap C = 4P_1,$$
$$\{x = 0\} \cap C = P_1 + 3P_2,$$

which implies that the divisor $P_1 - P_2$ is 3-torsion. Thus, since P_1 and P_2 are fixed by G, we have

$$\mathrm{Pic}^G(S) = \langle P_1, P_2 \rangle,$$

and, in particular, D is linearly equivalent to $2P_1$, $2P_2$ or P_1+P_2, which contradicts (1).

Finally, consider the case where G is isomorphic to $\mathbb{Z}_6$. Then the G-action is given by

$$[x:y:z] \mapsto \big[-\omega_6 x : -y : z\big],$$

where ω_6 is a primitive sixth root of unity. Set $P=[1:0:0]$. Then P is fixed by G, and

$$\{z=0\}\cap C = 4P.$$

Moreover, the intersection $\{x=0\}\cap C$ splits as a union of two G-orbits of length 2, which we denote by Σ_2 and Σ_2'. Then, by adjunction formula, we have

$$K_C \sim 4P \sim \Sigma_2+\Sigma_2'.$$

This gives $\mathrm{Pic}^G(S)=\langle P,\Sigma_2\rangle$. Moreover, it is not difficult to check that

$$3\Sigma_2+2P \sim 2K_C \sim 8P$$

whence $2P-\Sigma_2$ is a 3-torsion. Then D is linearly equivalent to $2P$, Σ_2, Σ_2', which contradicts (1). □

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